Dynamics of Condensation in Stochastic Particle Systems

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Plan

I **Real Space Condensation**

- Zero Range Process
- Factorised Steady State (**FSS**)
- Condensation and large deviations of sums of random variables

II **Explosive Condensation**

- 'Misanthrope' process
- Dynamics of condensation

III **Condensation through two constraints**

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References:

T Hanney and M.R. Evans, J. Phys. A (2005)
M. R. Evans, S. N. Majumdar and R. K. P. Zia J. Stat. Phys. (2006)
B. Waclaw and M. R. Evans, Phys. Rev. Lett. 108, 070601 (2012), J. Phys. A (2014)
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Stochastic Mass Transport Models

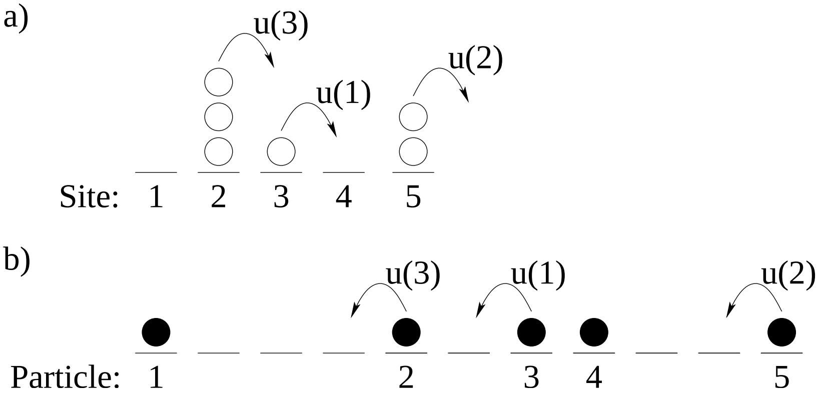
Conserved quantity "Mass" transferred stochastically from site to site of some lattice or network according to (local) dynamical rules

Examples

- Traffic — vehicular, granular, biological
- Rewiring networks — links transferred between nodes
- Econophysics — wealth transferred between agents
- Biophysics — polymerisation, actin filaments, transcription/translation

Dynamics leads to non-equilibrium stationary state

Zero-Range Process



a) "balls-in-boxes" picture

b) "Exclusion Process" picture

Motivation for ZRP

- Specific physical systems map onto ZRP

  e.g. polymer dynamics, shaken granular gases, traffic flow

- Effective description of dynamics involving exchange between domains

  e.g. phase separation dynamics

- Factorised Steady State (system of L sites and N particles)

$$\mathrm{P}(m_1 \ldots\ldots m_L) = \frac{1}{Z_{N,L}} f(m_1) \ldots f(m_L)\, \delta(\sum_i m_i - N)$$

where the single-site weight $f(m)$

$$f(m) = \prod_{n=1}^{m} \frac{1}{u(n)}$$

Factorised Stationary States

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Normalization (Nonequilibrium partition function)

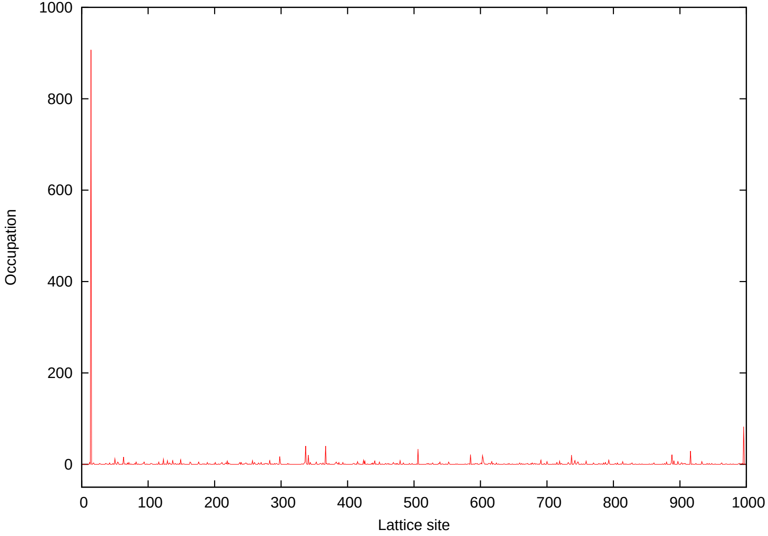
$$Z_{N,L} = \sum_{\{m_i=0\}}^{\infty} f(m_1)\dots f(m_L)\,\delta(\sum_j m_j - N)$$

Single-site mass distribution (Marginal distribution)

$$p(m) = f(m)\frac{Z_{N-m,L-1}}{Z_{N,L}}$$

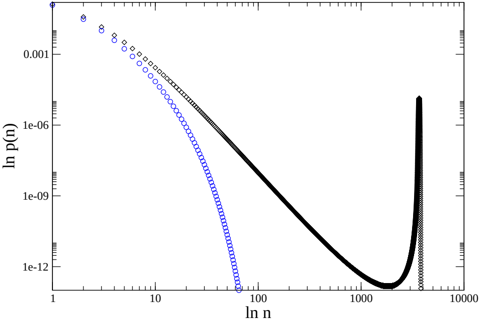
Real Space Condensation

Snapshot of ZRP $u(m) = 1 + \frac{3}{m}$ above critical density



Real Space Condensation

Single-site mass distribution in ZRP $u(m) = 1 + \frac{5}{m}$



below critical density ($\rho = \frac{N}{L}$)
above critical density (note condensate bump p_{bump})

Real Space Condensation

Grand Canonical Ensemble: $p_{gc}(m) = Az^m f(m) \quad z < 1 \quad z$ is fugacity

Constraint: $\displaystyle\sum_{m=0}^{\infty} m p_{gc}(m) = \rho \equiv \lim_{L,N\to\infty} \frac{N}{L}$

i.e. density $\rho(z)$ as function of z

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If $u(m) = 1 + \frac{\gamma}{m} \Rightarrow \boxed{f(m) \sim m^{-\gamma}}$

Then $z \to z^* = 1$ gives the max allowed value of density $\rho_{\max}$

$$\rho_{\max} \to \infty \qquad \text{if} \quad \gamma \leq 2$$
$$\rho_{\max} \to \rho_c \quad < \infty \qquad \text{if} \quad \gamma > 2$$

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$$
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\end{array}
$$

Thus for $\gamma > 2$ we have condensation if $\rho > \rho_c$

In condensed phase critical fluid coexists with condensate; grand canonical and canonical ensemble inequivalence

Nature of the Condensate: a large deviation effect

Canonical partition function: (computed in EMZ 2006)

$$Z_{N,L} = \sum_{\{m_i=0\}}^{\infty} \prod_{i}^{L} f(m_i)\delta\left(\sum_{j}^{L} m_j - N\right)$$

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Condensate shows up in a large deviation

of a sum of random variables when $N \gg \mu_1 L$ with $\sum_{m=0}^{\infty} m\, f(m) \equiv \mu_1 < \infty$.

For heavy-tailed $f(m)$, the event that $\sum_{i=1}^{L} m_i = N$ is most likely realised by one of m_i being $O(L)$ and the rest being $O(1)$

II Explosive Condensation

Consider Generalisation of ZRP to dependence on target site.

$u(m, n)$ is rate of hopping of particle from departure site containing m to target site containing n particles sometimes called 'misanthrope process' (Cocozza-Thivent 1985)

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A simple hopping rate which gives a factorised stationary state is

$$u(m,n) = [v(m) - v(0)]v(n)$$

then the single-site weight becomes

$$f(m) \propto \prod_{k=1}^{m} \frac{v(k-1)}{v(k) - v(0)}$$

For f to decay as $f(m) \sim m^{-\gamma}$ (for condensation) we now have several possible choices of asymptotic behaviour of $v(m)$

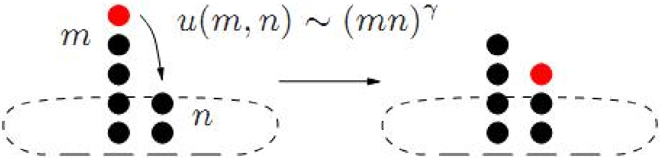
$$v(m) \simeq 1 - \frac{\alpha}{m} \qquad \text{'ZRP like' } (\gamma \text{ is function of } \alpha \text{ and } v(0))$$

$$v(m) \sim m^{\gamma} \qquad \text{'explosive'}$$

Explosive dynamics

$$u(m, n) = [v(m) - v(0)]v(n)$$

with $v(m) = (\epsilon + m)^\gamma$ and $\epsilon > 0$

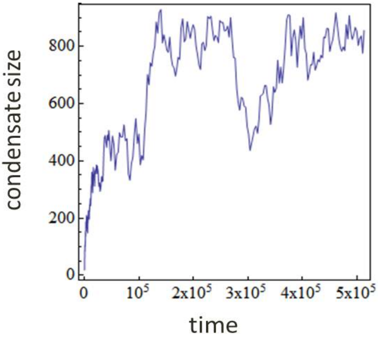


Get condensation for $\gamma > 2$.

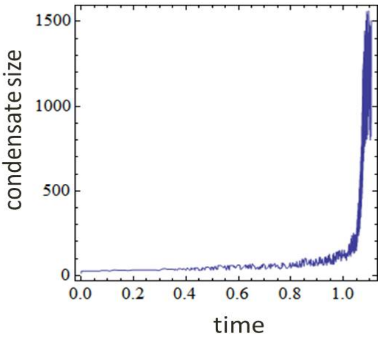
Contrasting Dynamics

Both choices (ZRP-like, explosive) generate same stationary state (condensed) but the dynamics are very different:

zero-range process



our model



$T_{SS} \sim L^2$

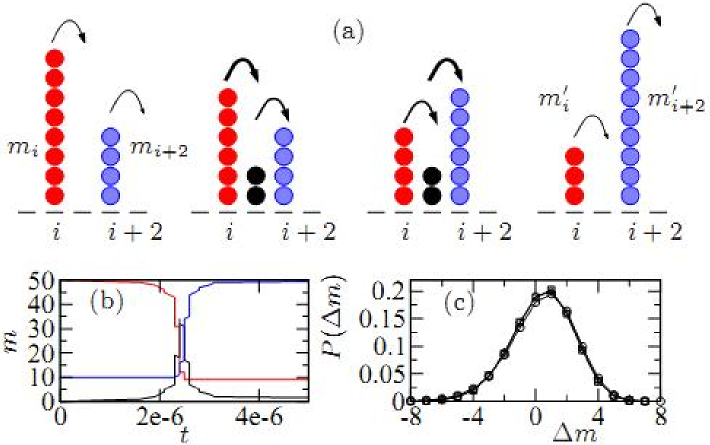
$T_{SS} \sim (\ln L)^{1-\gamma} \quad (?)$

Condensate stationary

Condensate moves with slinky motion
Speed of condensate $v(m) \sim m^\gamma$

Explosive Dynamics

Scattering collisions between two condensates



- Almost elastic scattering
- Larger condensate picks up mass

III Condensation induced by two constraints

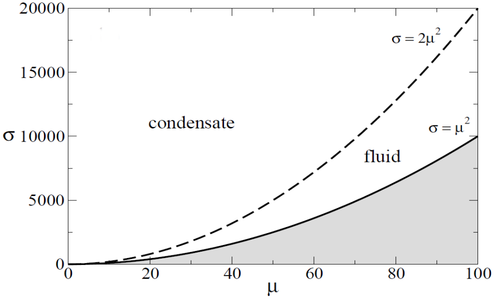
Consider a continuous mass system with two constraints:

$$N = \rho L = \sum_i m_i \qquad V = \sigma L = \sum_i m_i^{1/p}$$

Local dynamics exchanging mass and respecting constraints leads to factorised steady state (Iubini, Politi and Politi 2014).

Condensation now occurs even for light-tailed f e.g. $f(m) = a \exp -am$.
(Szavits-Nossan, Evans and Majumdar 2014)

Condensate is now $O(L^p)$ for $p < 1$.



Conclusions

- Real space condensation — ubiquitous dynamical phase transition in variety of contexts

  Analysable within ZRP FSS
- Understanding in terms of large deviations of sum of random variables
- Explosive Condensation has same stationary state as ZRP but relaxation time $T \sim (\ln L)^{1-\gamma}$ vanishes for large L
- Two constraint problem reminiscent of DNLSE