Optical wave thermalization

→ **in the framework of optical wave turbulence**
→ **nonequilibrium kinetic formulation**

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Theory

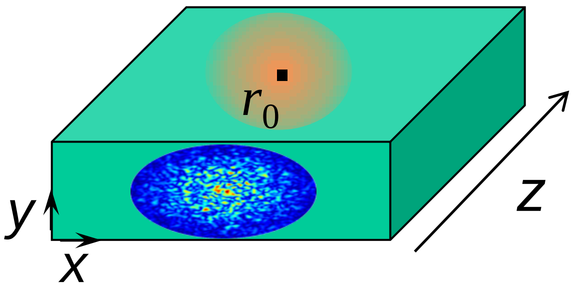
J. Garnier (ENS-Paris)
S. Rica (Un. Ibanez, Chile)

Experiments & Simulations

G. Xu, B. Kibler, G. Millot (ICB)
S. Trillo (Un. Ferrara)
C. Michel, P. Aschieri (LPMC, Nice)
R. Kaiser (INLN, Nice)
D. Faccio, D. Vocke, T. Roger (HW, Edinburgh)
S. Turitsyn, D. Churkin (Aston Un.)

Nonequilibrium Processes at Negative Temperature
Glasgow – 23-24 oct. 2014

Spatial NLS eqn



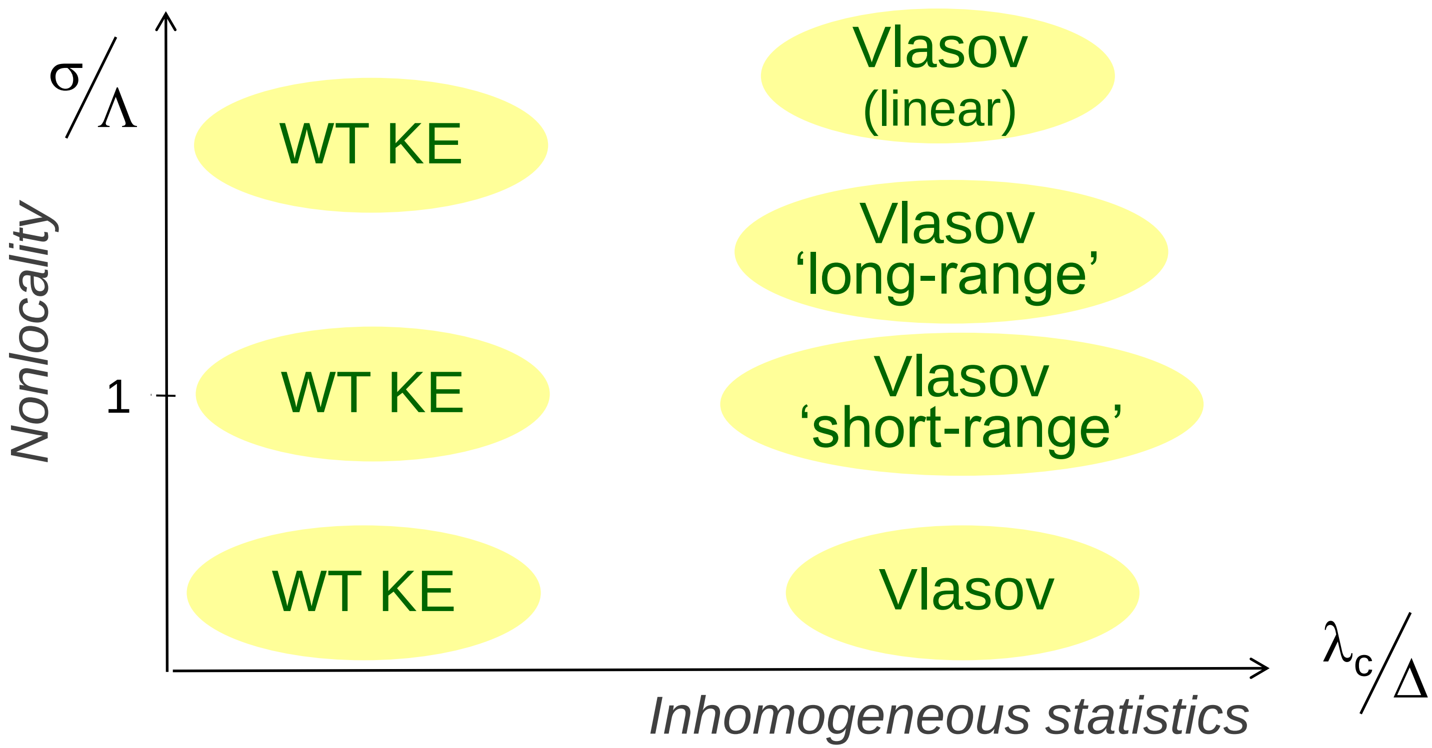
$$i\partial_z\psi = -\beta\,\nabla_\perp^2\psi + g\,\psi\int U(r')\,|\psi|^2(r-r')\,dr'$$

$U(x,y)$ nonlocal response (real, even): σ

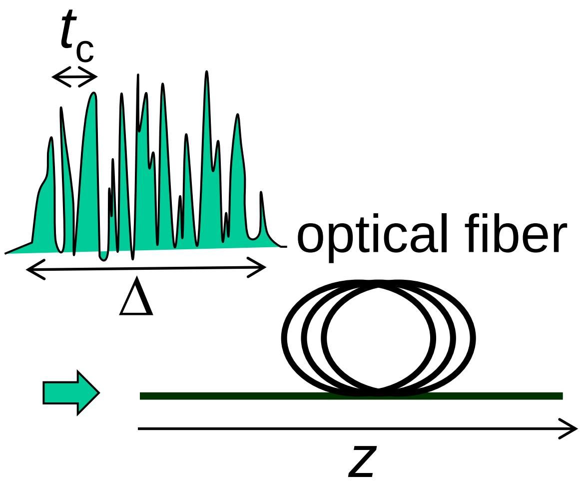
$\Lambda = \sqrt{\beta/(g|\psi|^2)}$: healing length

Δ : scale of inhomogeneous statistics

λ_c : correlation length



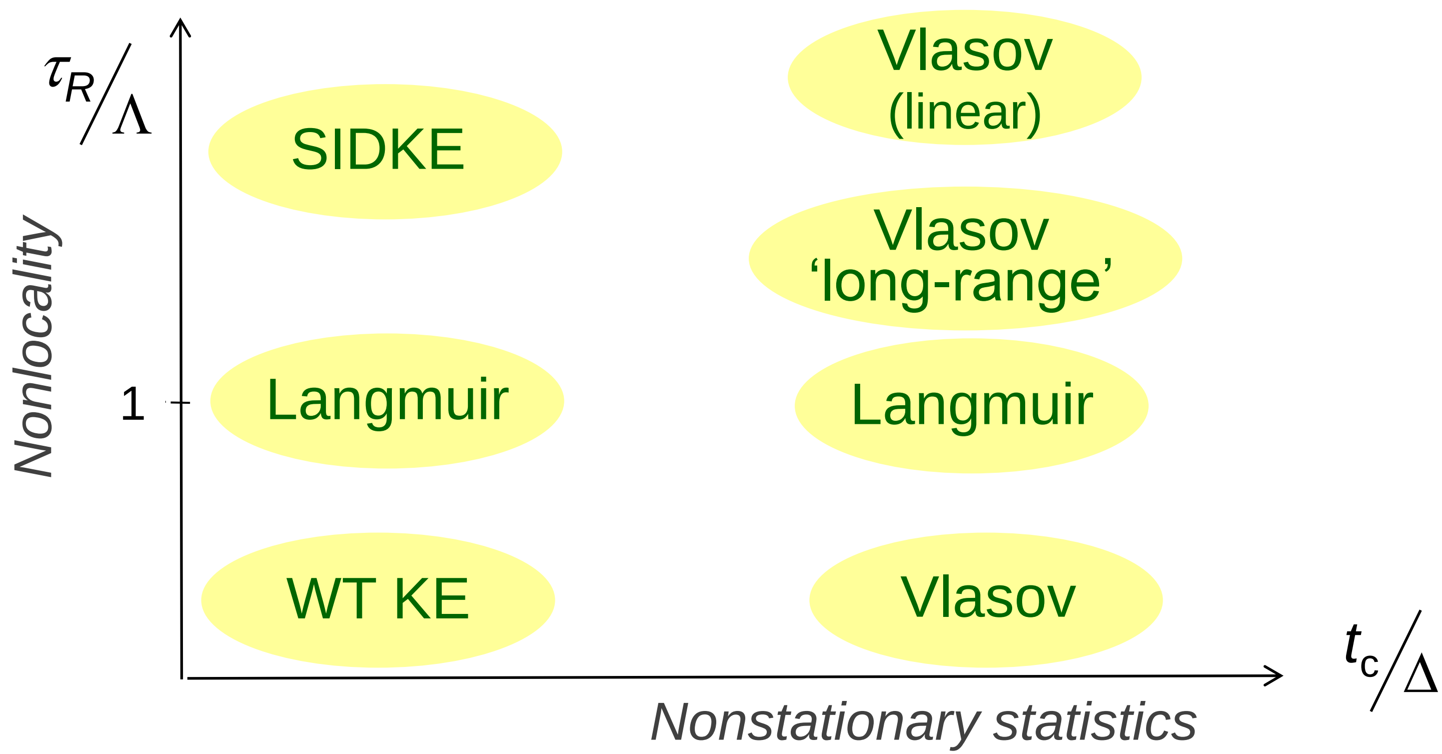
Temporal NLS eqn



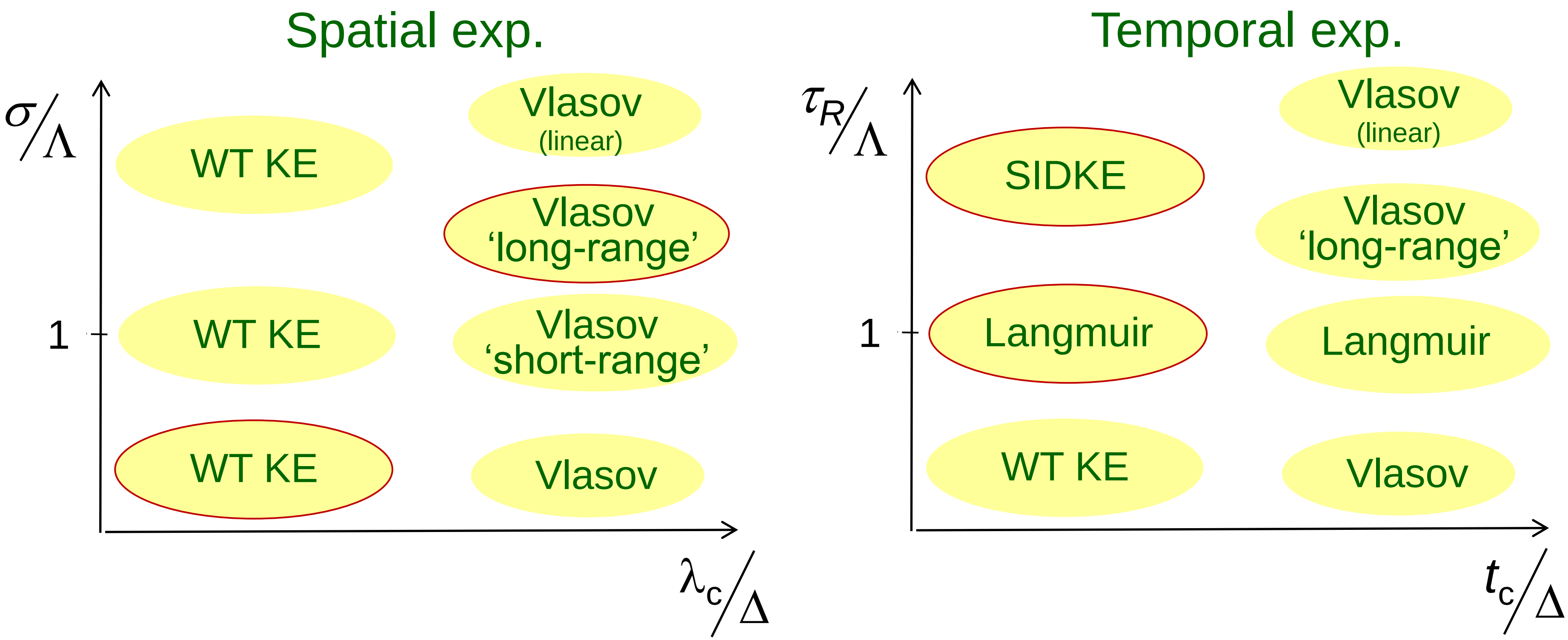
$$i\partial_z\psi = -\beta\,\partial_{tt}\psi + g\,\psi\int R(t')\,|\psi|^2(t-t')\,dt'$$

$R(t)$ is causal: $R(t)=0$ for $t<0$

- $\Lambda = \sqrt{\beta/(g|\psi|^2)}$: healing time
- Δ : scale of inhomogeneous statistics
- t_c : correlation time



Optical wave thermalization…?



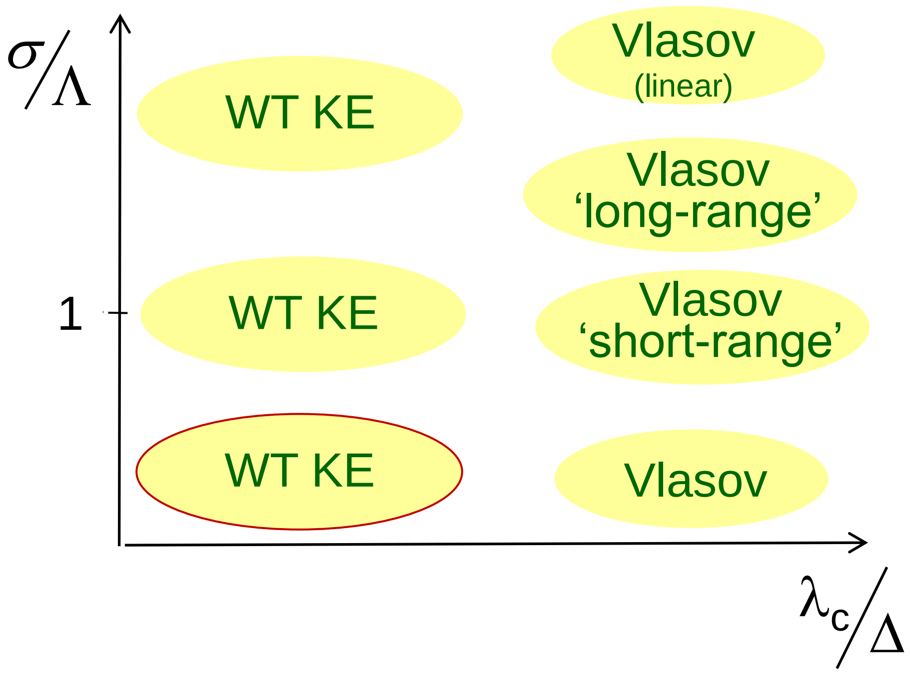
Optical Wave Turbulence -- *Review*

Picozzi, Garnier, Hansson, Suret, Randoux, Millot, Christodoulides -- Physics Reports, 2014

Optical wave turbulence

1.- WT kinetic equation: Thermalization & Condensation:

- **Wave turbulence in a trap**
- **Breakdown of thermalization**



2.- Long-range Vlasov Turbulence:

- Breakdown of thermalization

3.- Weak Langmuir Turbulence:

- Breakdown of thermalization

‘Condensation’ of classical waves

Renewed interest since:

- Condensation in mode-locked lasers (Fischer, Gat, Gordon…)
- Condensation in random lasers (Conti)
- Spectral condensation in long fibers (Turitsyn, Falkovich…)
- Photon condensation in optical microcavities (Weitz group)
- Polariton condensates (RMP Carusotto, Ciuti)
- Lasing or Condensation ? (Carusotto, Ciuti, Keeling, Kirton…)

⟶ ***Here:* ‘Wave Condensation’**

- **without gain & without cavity**
- **classical limit**

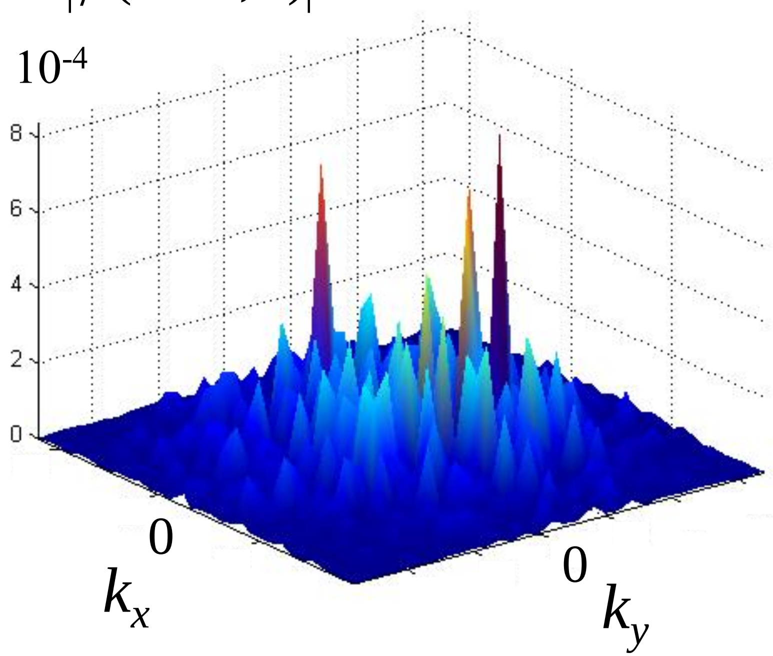
Reminder: Wave condensation

$$i\partial_z \psi = -\nabla^2 \psi + |\psi|^2 \psi$$

- Defocusing
- **Periodic boundary conditions !**

- Hamiltonian
- Reversible ($z \rightarrow -z$)

$|\tilde{\psi}(z=0,k)|^2$



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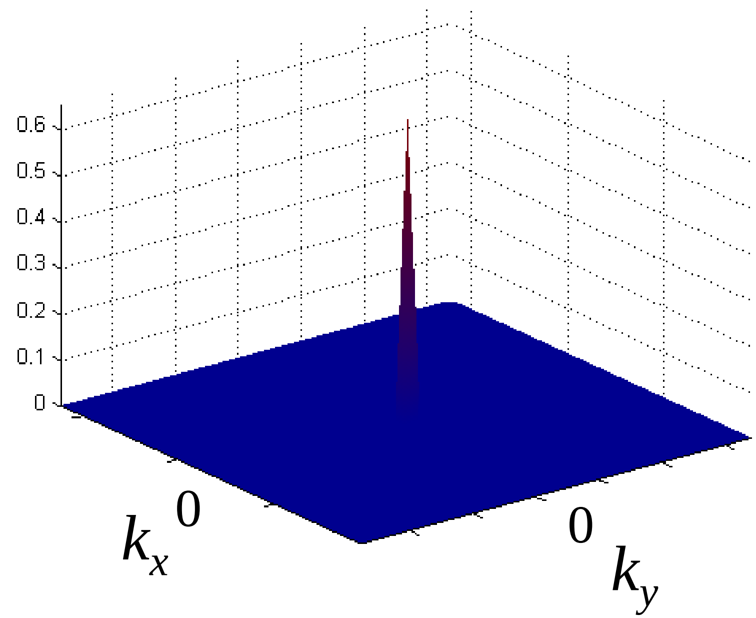
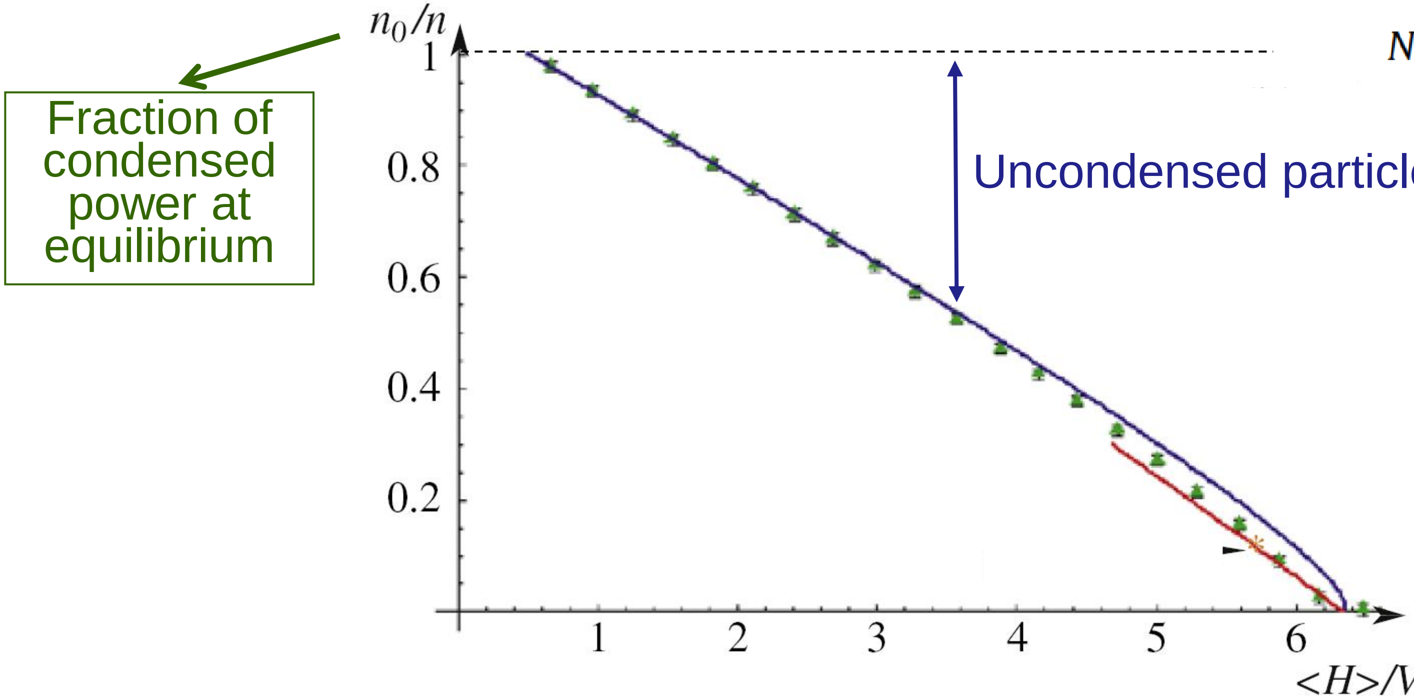


Condensation results from the natural thermalization to equilibrium

Dyachenko et al., -- Physica D (1992)
Davis, Morgan, Burnett – PRL (2002)
Connaughton, Josserand, Picozzi, Pomeau, Rica -- PRL (2005)
Onorato, Nazarenko -- Physica D (2006)
Düring, Picozzi, Rica -- Physica D (2009)

Reminder: Fraction of condensed power

$$N = \int |\psi|^2 \mathrm{d}^D \boldsymbol{x} = const$$



$$\frac{\langle H \rangle}{V} = (n - n_0) \frac{\sum_{\boldsymbol{k}}' 1}{\sum_{\boldsymbol{k}}' \frac{1}{k^2}} + a \left(n^2 - \frac{1}{2} n_0^2 \right)$$

Small Cond. Amp.: WT

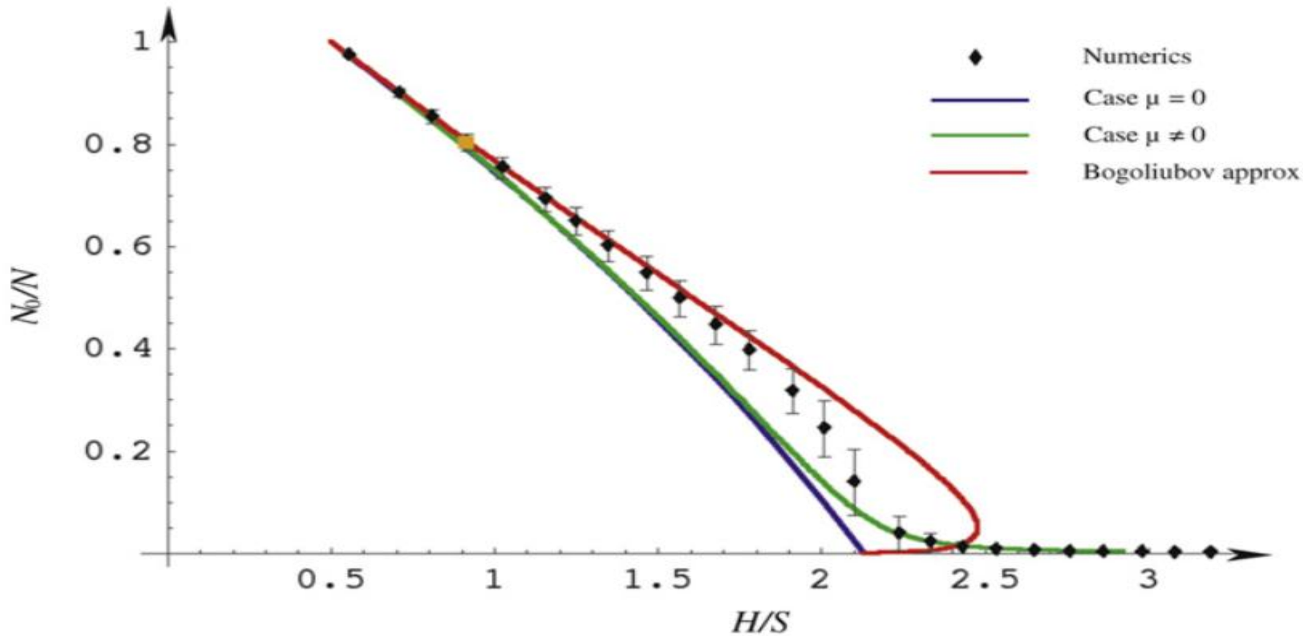
$$\langle H \rangle / V = a \frac{n^2}{2} + \frac{a}{2} (n - n_0)^2 + (n - n_0) \frac{\sum_{\boldsymbol{k}}' 1}{\sum_{\boldsymbol{k}}' \frac{k^2 + a n_0}{k^4 + 2 a n_0 k^2}}$$

High Cond. Amp.: Bogoliubov

Connaughton, Josserand, Picozzi, Pomeau, Rica -- PRL (2005)
During, Picozzi, Rica -- Physica D (2009)

Reminder: Wave condensation

Nonlocal nonlinearity: $$i\partial_z A(z,\mathbf{r}) = -\nabla^2 A + A\int V(\mathbf{r}-\mathbf{r}')\,\left|A(z,\mathbf{r}')\right|^2\, d\mathbf{r}'$$



$$\begin{cases} \dfrac{\langle H(\mu)\rangle}{S} = (1-n_0)\dfrac{\sum'_{\mathbf{k}} \dfrac{k^2+n_0\hat{V}_{\mathbf{k}}}{k^2-\mu}}{\sum'_{\mathbf{k}} \dfrac{1}{k^2-\mu}} + \dfrac{1}{2} + \dfrac{(1-n_0)^2}{2}\dfrac{\sum'_{\mathbf{k}_1,\mathbf{k}_2} \dfrac{\hat{V}_{\mathbf{k}_1-\mathbf{k}_2}}{(k_1^2-\mu)(k_2^2-\mu)}}{\left(\sum'_{\mathbf{k}} \dfrac{1}{k^2-\mu}\right)^2} \\ \dfrac{N_0(\mu)}{N} = \dfrac{1}{-\mu}\dfrac{1}{\sum_{\mathbf{k}} \dfrac{1}{k^2-\mu}} \end{cases}$$

Small Cond. Amp.: WT

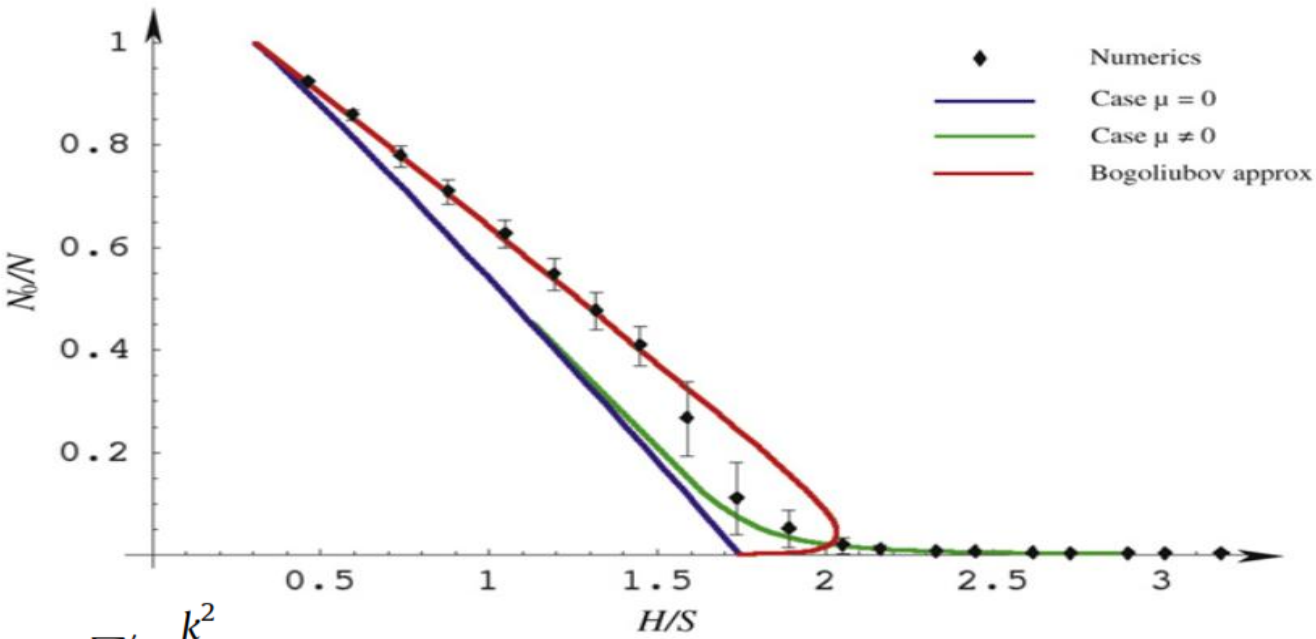
$$\frac{\langle H(\mu)\rangle}{S} = (1-n_0)\frac{\sum'_{\mathbf{k}} \dfrac{k^2+n_0\hat{V}_{\mathbf{k}}}{k^2}}{\sum'_{\mathbf{k}} \dfrac{1}{k^2}} + \frac{1}{2} + \frac{(1-n_0)^2}{2}\frac{\sum'_{\mathbf{k}_1,\mathbf{k}_2} \dfrac{\hat{V}_{\mathbf{k}_1-\mathbf{k}_2}}{k_1^2\,k_2^2}}{\left(\sum'_{\mathbf{k}} \dfrac{1}{k^2}\right)^2}$$

High Cond. Amp.: Bogoliubov

Picozzi, Rica – OC (2012)

Reminder: Wave condensation

Saturable nonlinearity: $$i\partial_z A = -\nabla^2 A + g(|A|^2)\,A \qquad g(|A|^2) = \frac{\rho |A|^2}{1+\rho|A|^2}$$



$$\begin{cases} \dfrac{\langle H(\mu)\rangle}{S} = (1-n_0)\dfrac{\sum_k' \frac{k^2}{k^2-\mu}}{\sum_k' \frac{1}{k^2-\mu}} + \dfrac{\langle U(n_0)\rangle}{S} \\[2ex] \dfrac{N_0(\mu)}{N} = \dfrac{1}{-\mu}\dfrac{1}{\sum_k \frac{1}{k^2-\mu}} \end{cases}$$

$$\begin{cases} \dfrac{\langle U(n_0)\rangle}{S} = 1 - \dfrac{1}{\rho(1-n_0)}\displaystyle\int_0^\infty \ln(1+\rho I)\exp\left(-\frac{I+n_0}{1-n_0}\right) I_0\left(\frac{2\sqrt{n_0 I}}{1-n_0}\right) dI \\[2ex] f_I(I) = \theta(I)\dfrac{\exp\left(-\frac{I+n_0}{1-n_0}\right)}{1-n_0} I_0\left(\dfrac{2\sqrt{n_0 I}}{1-n_0}\right) \end{cases}$$

$$\frac{\langle H\rangle}{S} = \frac{\langle U(n_0)\rangle}{S} - \frac{\rho n_0(1-n_0)}{(1+\rho n_0)^2} + \frac{(1-n_0)(N_* - 1)}{\sum_k' \frac{k^2+\rho n_0/(1+\rho n_0)^2}{\omega_B^2(k)}} \qquad \omega_B(k) = \sqrt{k^4 + \frac{2\rho n_0}{(1+\rho n_0)^2}k^2}$$

Picozzi, Rica – OC (2012)

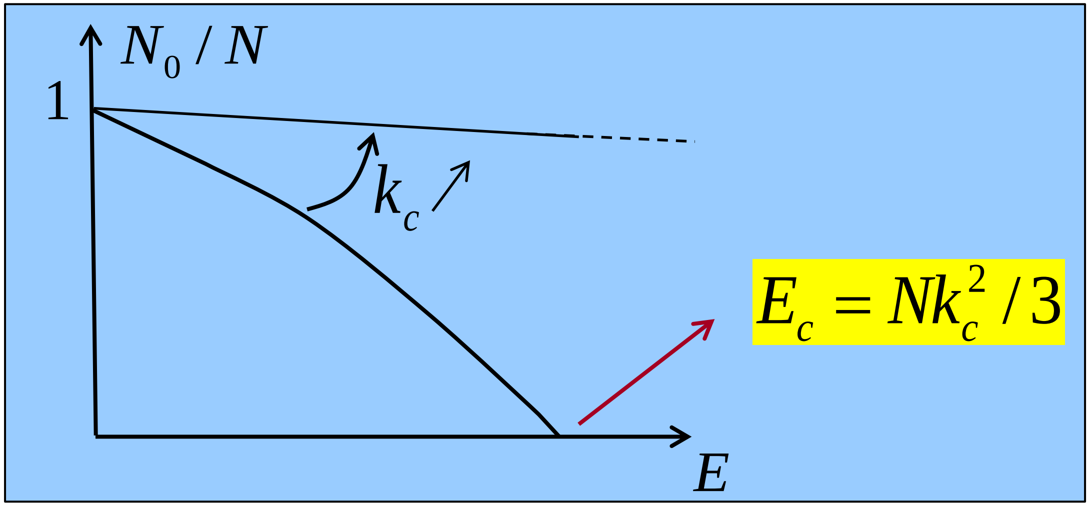
Condensation of classical waves…. Problem

$$\begin{cases} i\partial_z\psi = -\nabla^2\psi + |\psi|^2\psi \\ N = \int n_k dk \\ E = \int k^2 n_k dk \end{cases} \xrightarrow{\text{WT}} \begin{cases} \text{Rayleigh-Jeans} \\ n_k^{eq} = \dfrac{T}{k^2 - \mu} \end{cases}$$

$k_c = \pi / dx$: UV catastrophe

$$3\text{D} \begin{cases} N = 4\pi T k_c \left[1 - \dfrac{\sqrt{-\mu}}{k_c} \text{Arctg}\left(\dfrac{k_c}{\sqrt{-\mu}} \right) \right] \\ \mu \to 0 \ \text{ for } \ T = T_c \neq 0 \end{cases}$$

$$2\text{D} \begin{cases} N = \pi T \ \text{Log}(1 - k_c^2 / \mu) \\ \mu \to 0 \ \text{ for } \ T = 0 \end{cases}$$

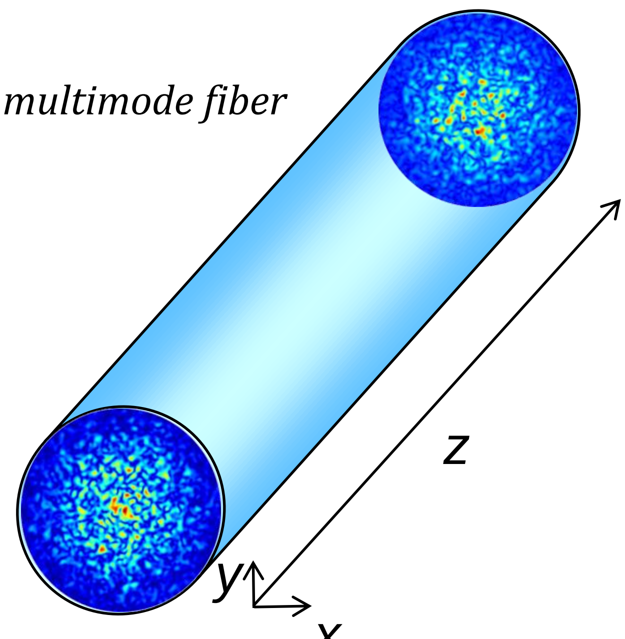


Davis, Morgan, Burnett -- PRL (2002); PRA (2002)
Connaughton, Josserand, Picozzi, Pomeau, Rica -- PRL (2005)
During, Picozzi, Rica -- Physica D (2009)

Wave turbulence & wave condensation in a trap *V(r)*

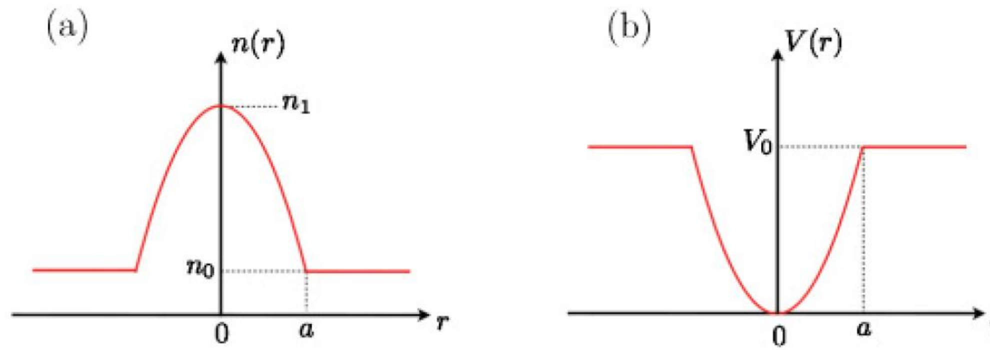
$$i\partial_z \psi = -\alpha \nabla^2 \psi + V(\mathbf{r})\psi + \gamma |\psi|^2 \psi$$

(D = 2)



$\vec{r} = (x, y)$

$$V(\mathbf{r}) = k_0 \left(\frac{n_1^2 - n^2(\mathbf{r})}{2n_0} \right)$$



$$N = \int |\psi|^2 d\mathbf{r}$$

$$H = \underbrace{\int \alpha |\nabla \psi|^2 d\mathbf{r} + \int V(\mathbf{r}) |\psi|^2 d\mathbf{r}}_{\text{Linear, } E} + \underbrace{\frac{\gamma}{2} \int |\psi|^4 d\mathbf{r}}_{\text{Nonlinear, } U}$$

WT kinetic eqn in a trap *V(r)*

$$i\partial_z \psi = -\alpha \nabla^2 \psi + V(\mathbf{r})\psi + \gamma |\psi|^2 \psi \qquad \psi(\mathbf{r}, z) = \sum_m c_m(z) u_m(\mathbf{r}) \exp(-i\beta_m z)$$

$$i\partial_z a_m = \beta_m a_m + \gamma \sum_{p,q,s} W_{mpqs} a_p a_q^* a_s \qquad a_m(z) = c_m(z)\exp(-i\beta_m z)$$

$$W_{mpqs} = \int u_m^*(\mathbf{r}) u_p(\mathbf{r}) u_q^*(\mathbf{r}) u_s(\mathbf{r}) d\mathbf{r} \qquad n_m(z) = \left\langle \left| \int \psi(\boldsymbol{r}, z)\, u_m^*(\boldsymbol{r})\, d\boldsymbol{r} \right|^2 \right\rangle = \langle |c_m(z)|^2 \rangle$$

Continuous limit: $V_0/\beta_0 \gg 1$ & Weakly nonlinear regime

WT theory
- Zakharov, L'vov, Falkovich, *Kolmogorov Spectra of Turbulence I* (1992)
- Nazarenko, *Wave Turbulence* (2011)

$$\partial_z \tilde{n}_\kappa = \frac{4\pi\gamma^2}{\beta_0^6} \iiint d\kappa_1 d\kappa_2 d\kappa_3 \delta(\tilde{\beta}_{\kappa_1} + \tilde{\beta}_{\kappa_3} - \tilde{\beta}_{\kappa_2} - \tilde{\beta}_\kappa) \times |\widetilde{W}_{\kappa\kappa_1\kappa_2\kappa_3}|^2 \tilde{n}_\kappa \tilde{n}_{\kappa_1} \tilde{n}_{\kappa_2} \tilde{n}_{\kappa_3} (\tilde{n}_\kappa^{-1} + \tilde{n}_{\kappa_2}^{-1} - \tilde{n}_{\kappa_1}^{-1} - \tilde{n}_{\kappa_3}^{-1}) + \frac{8\pi\gamma^2}{\beta_0^2} \int d\kappa_1 \delta(\tilde{\beta}_{\kappa_1} - \tilde{\beta}_\kappa) |\tilde{U}_{\kappa\kappa_1}(\tilde{n})|^2 (\tilde{n}_{\kappa_1} - \tilde{n}_\kappa)$$

$$\tilde{U}_{\kappa\kappa_1}(\tilde{n}) = \frac{1}{\beta_0^2} \int d\kappa'\, \widetilde{W}_{\kappa\kappa_1\kappa'\kappa'} \tilde{n}_{\kappa'}$$

$$\tilde{\beta}_\kappa = \beta_{[k/\beta_0]}$$

$$\tilde{n}_\kappa(z) = n_{[k/\beta_0]}(z)$$

$$\widetilde{W}_{\kappa\kappa_1\kappa_2\kappa_3} = W_{[k/\beta_0][k_1/\beta_0][k_2/\beta_0][k_3/\beta_0]}$$

$$\boldsymbol{\kappa} = \beta_0 (m_x, m_y)$$

WT in a trap *V(r)* – *multimode fiber*

$$\partial_z \tilde{n}_\kappa = \frac{4\pi\gamma^2}{\beta_0^6} \iiint d\kappa_1 d\kappa_2 d\kappa_3 \delta(\tilde{\beta}_{\kappa_1} + \tilde{\beta}_{\kappa_3} - \tilde{\beta}_{\kappa_2} - \tilde{\beta}_\kappa)$$
$$\times \left|\widetilde{W}_{\kappa\kappa_1\kappa_2\kappa_3}\right|^2 \tilde{n}_\kappa \tilde{n}_{\kappa_1} \tilde{n}_{\kappa_2} \tilde{n}_{\kappa_3} (\tilde{n}_\kappa^{-1} + \tilde{n}_{\kappa_2}^{-1} - \tilde{n}_{\kappa_1}^{-1} - \tilde{n}_{\kappa_3}^{-1})$$
$$+ \frac{8\pi\gamma^2}{\beta_0^2} \int d\kappa_1 \delta(\tilde{\beta}_{\kappa_1} - \tilde{\beta}_\kappa) \left|\widetilde{U}_{\kappa\kappa_1}(\tilde{n})\right|^2 (\tilde{n}_{\kappa_1} - \tilde{n}_\kappa)$$

$$\widetilde{U}_{\kappa\kappa_1}(\tilde{n}) = \frac{1}{\beta_0^2} \int d\kappa' \, \widetilde{W}_{\kappa\kappa_1\kappa'\kappa'} \tilde{n}_{\kappa'}$$

$$W_{mpqs} = \int u_m^*(\mathbf{r}) u_p(\mathbf{r}) u_q^*(\mathbf{r}) u_s(\mathbf{r}) d\mathbf{r}$$
$$\widetilde{W}_{\kappa\kappa_1\kappa_2\kappa_3} = W_{[k/\beta_0][k_1/\beta_0][k_2/\beta_0][k_3/\beta_0]}$$

Generalization of the conventional WT eqn: *V(r) = 0*

Plane-wave expansion & periodic boundary conditions:

$$u_{m_x,m_y}(\boldsymbol{r}) = \frac{1}{L} \exp[2i\pi(m_x x + m_y y)/L]$$

$$|\tilde{W}_{\boldsymbol{\kappa}\boldsymbol{\kappa}_1\boldsymbol{\kappa}_2\boldsymbol{\kappa}_3}|^2 = \frac{(2\pi)^2}{L^6} \delta(\boldsymbol{k}_1 + \boldsymbol{k}_3 - \boldsymbol{k}_2 - \boldsymbol{k})$$

$$\partial_z \tilde{n}_{\boldsymbol{k}}(z) = \frac{4\pi\gamma^2}{(2\pi)^2} \iiint d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \delta\left(\alpha\left(k_1^2 + k_3^2 - k_2^2 - k^2\right)\right) \delta(\boldsymbol{k}_1 + \boldsymbol{k}_3 - \boldsymbol{k}_2 - \boldsymbol{k}) \mathcal{N}(\tilde{\mathbf{n}})$$

$$\mathcal{N}(\tilde{\mathbf{n}}) = \tilde{n}_{\boldsymbol{k}} \tilde{n}_{\boldsymbol{k}_1} \tilde{n}_{\boldsymbol{k}_2} \tilde{n}_{\boldsymbol{k}_3} \left(\tilde{n}_{\boldsymbol{k}}^{-1} + \tilde{n}_{\boldsymbol{k}_2}^{-1} - \tilde{n}_{\boldsymbol{k}_1}^{-1} - \tilde{n}_{\boldsymbol{k}_3}^{-1}\right)$$

$$\mathbf{P} = \int \boldsymbol{k} \, \tilde{n}_{\boldsymbol{k}} \, d\boldsymbol{k}$$

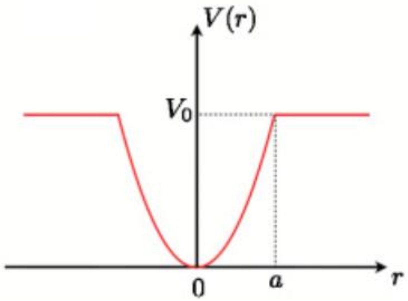
Aschieri, Garnier, Michel, Doya, Picozzi -- PRA (2011)

WT in a trap *V(r)*

$$\partial_z \tilde{n}_\kappa = \frac{4\pi\gamma^2}{\beta_0^6} \iiint d\kappa_1 d\kappa_2 d\kappa_3 \delta(\tilde{\beta}_{\kappa_1} + \tilde{\beta}_{\kappa_3} - \tilde{\beta}_{\kappa_2} - \tilde{\beta}_\kappa)$$
$$\times \left|\widetilde{W}_{\kappa\kappa_1\kappa_2\kappa_3}\right|^2 \tilde{n}_\kappa \tilde{n}_{\kappa_1} \tilde{n}_{\kappa_2} \tilde{n}_{\kappa_3} (\tilde{n}_\kappa^{-1} + \tilde{n}_{\kappa_2}^{-1} - \tilde{n}_{\kappa_1}^{-1} - \tilde{n}_{\kappa_3}^{-1})$$
$$+ \frac{8\pi\gamma^2}{\beta_0^2} \int d\kappa_1 \delta(\tilde{\beta}_{\kappa_1} - \tilde{\beta}_\kappa) \left|\widetilde{U}_{\kappa\kappa_1}(\tilde{n})\right|^2 (\tilde{n}_{\kappa_1} - \tilde{n}_\kappa)$$

$$\widetilde{U}_{\kappa\kappa_1}(\tilde{n}) = \frac{1}{\beta_0^2} \int d\kappa' \, \widetilde{W}_{\kappa\kappa_1\kappa'\kappa'} \tilde{n}_{\kappa'}$$

$$N = \beta_0^{-2} \int d\boldsymbol{\kappa} \tilde{n}_{\boldsymbol{\kappa}}$$
$$E = \beta_0^{-2} \int d\boldsymbol{\kappa} \tilde{\beta}_{\boldsymbol{\kappa}} \tilde{n}_{\boldsymbol{\kappa}}$$
$$\mathcal{S}(z) = \beta_0^{-2} \int d\boldsymbol{\kappa} \ln(\tilde{n}_{\boldsymbol{\kappa}})$$: *H*-theorem



$$\tilde{n}_\kappa^{eq} = \frac{T}{\tilde{\beta}_\kappa - \mu}$$

$$\tilde{\varepsilon}_\kappa = \tilde{\beta}_\kappa \tilde{n}_\kappa^{eq} \sim T$$

Energy equipartition

Wave condensation in the thermodynamic limit 2D (parabolic potential):

$$N = \frac{T}{\beta_0^2}\left[V_0 - \tilde{\mu}\ln\left(\frac{-\tilde{\mu}}{V_0 - \tilde{\mu}}\right)\right] \qquad \tilde{\mu} = \mu - \beta_0$$

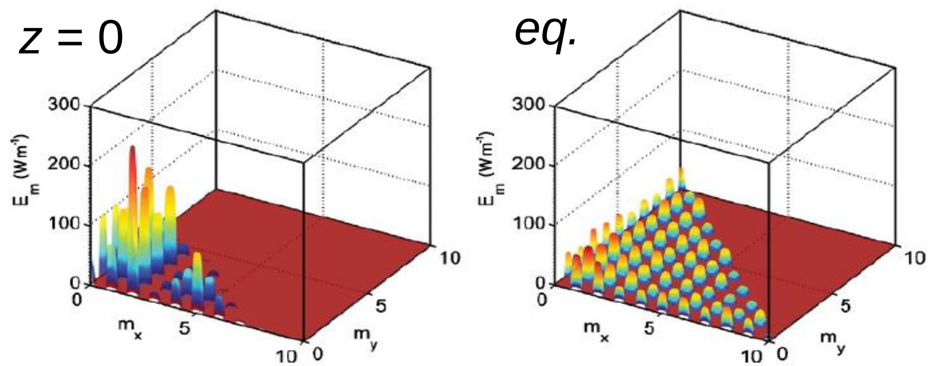
WT in a parabolic trap *(graded index fiber)*

$$i\partial_z\psi = -\alpha\nabla^2\psi + V(\mathbf{r})\psi + \gamma|\psi|^2\psi$$

- *Energy equipartition:*

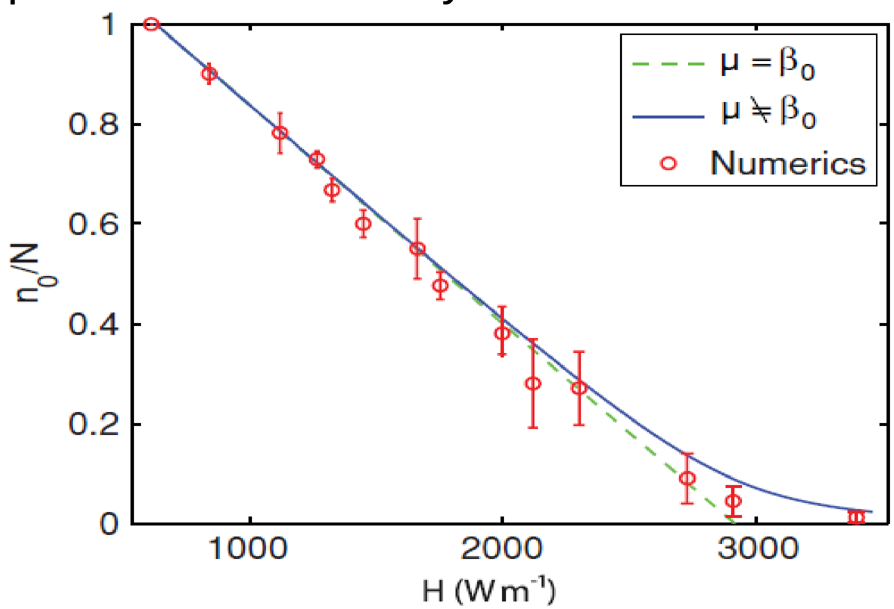
$$n_m(z) = \langle \left| \iint \psi(\mathbf{r},z)u_m^*(\mathbf{r})d\mathbf{r} \right|^2 \rangle = \langle |c_m(z)|^2 \rangle$$

$$\beta_m \simeq \beta_0(m_x + m_y + 1)$$



- *Parabolic trap:* Wave condensation takes place in the thermodynamic limit in 2D

$$E_c = E_0 + NV_0/2 = \frac{NV_0}{2}\left(1 + \frac{2\beta_0}{V_0}\right)$$



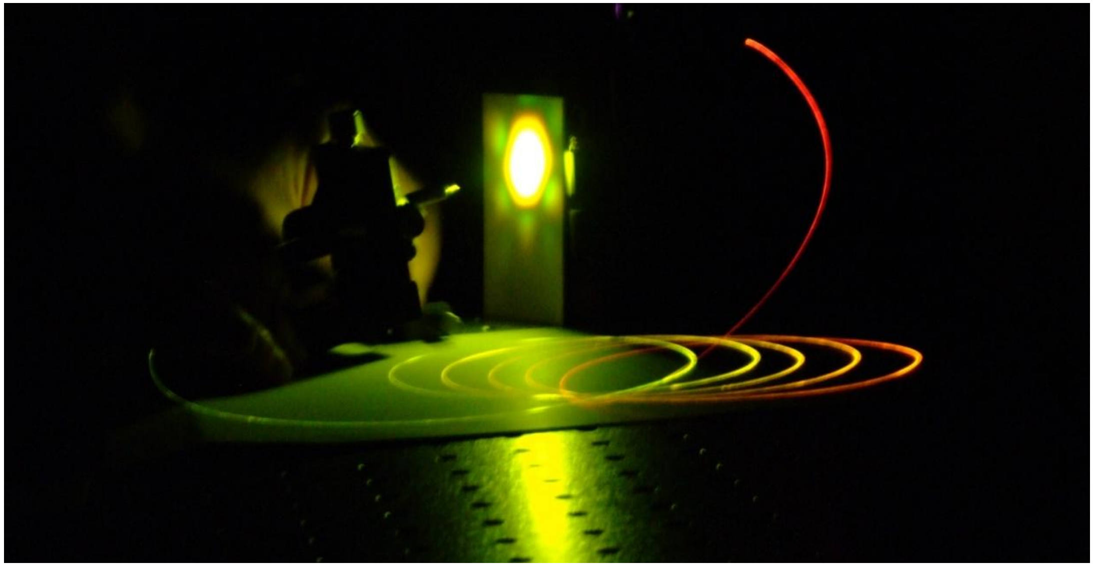
$$\begin{cases} \dfrac{n_0}{N}(\mu) = \dfrac{1}{-\mu\sum_m \dfrac{1}{\beta_m - \beta_0 - \mu}} \\ \langle H\rangle(\mu) = N\dfrac{\sum_m \dfrac{\beta_m}{\beta_m - \beta_0 - \mu}}{\sum_m \dfrac{1}{\beta_m - \beta_0 - \mu}} + \langle U\rangle(\mu) \\ \langle U\rangle(\tilde{\mu}) = \gamma\left[\dfrac{\rho}{2}n_0^2 - 2n_0^2\tilde{\mu}\int |u_0|^2(r)\sum_m{}' \dfrac{|u_m(\boldsymbol{r})|^2}{\beta_m - \beta_0 - \tilde{\mu}}d\boldsymbol{r} + n_0^2\tilde{\mu}^2\int\left(\sum_m{}' \dfrac{|u_m(\boldsymbol{r})|^2}{\beta_m - \beta_0 - \tilde{\mu}}\right)^2 d\boldsymbol{r}\right] \end{cases}$$

Aschieri, Garnier, Michel, Doya, Picozzi -- PRA (2011)

Thermalization for *D* = 2 or *D* = 3

⟶ What about *D* = 1…?

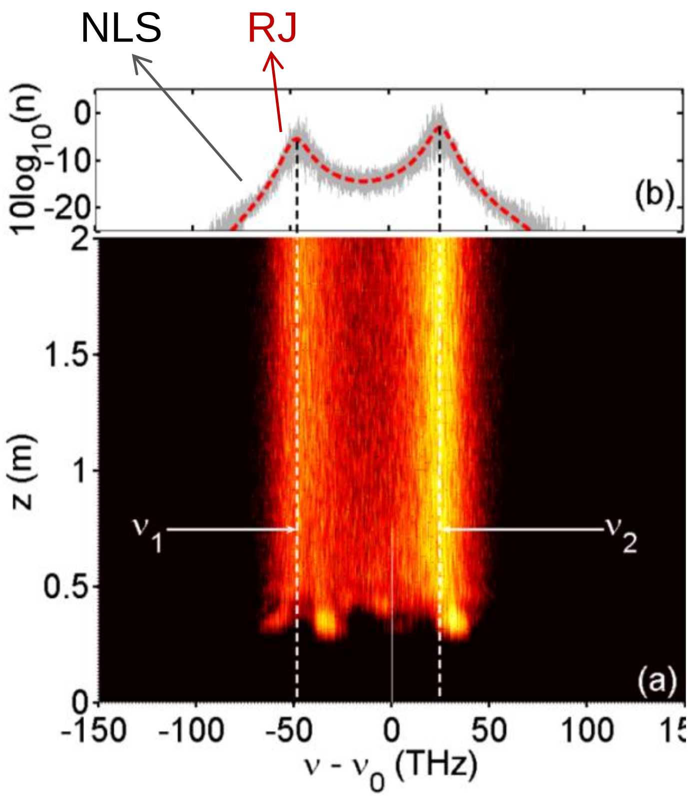
⟶ Supercontinuum generation in a photonic crystal fiber



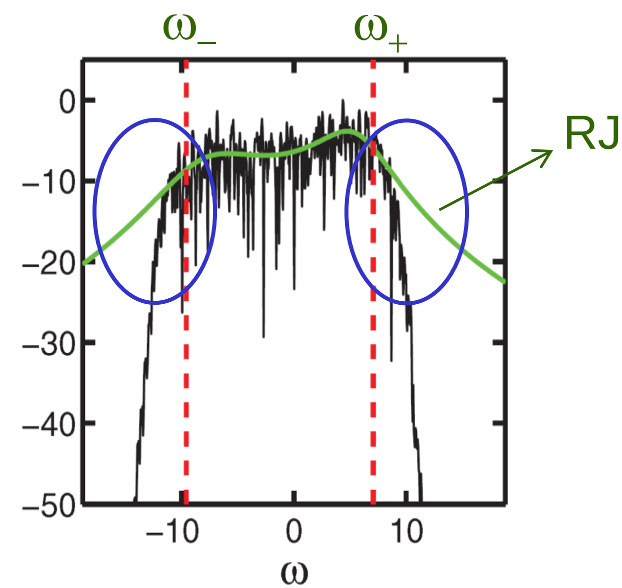
Optical wave thermalization?

Breakdown of thermalization for D = 1…?

→ D = 1: Supercontinuum generation in a photonic crystal fiber



$$n^{RJ}(\omega) = \frac{T}{k(\omega) + \lambda\omega - \mu} \quad (N, M, E)$$



$$-i\frac{\partial\psi}{\partial z} = \sum_{j\geq 2}^{m} \frac{i^j\beta_j}{j!}\frac{\partial^j\psi}{\partial t^j} + \gamma|\psi|^2\psi + i\gamma\tau_s\frac{\partial(|\psi|^2\psi)}{\partial t}$$

→ *WT theory provides a nonequilibrium kinetic description of SC generation*

See Review: Optical Wave Turbulence
Picozzi et al., Physics Reports (2014)

D = 1: Breakdown of thermalization…

→ The Fermi-Pasta-Ulam Problem: A Status Report,
Ed. by G. Gallavotti, Lecture Notes in Physics (Springer, New York, 2007)

→ Light propagation in a photonic crystal fiber

a) 2 ZDWs: Truncated RJ

$$i\partial_z\psi = -\partial_t^2\psi + i\alpha\partial_t^3\psi + \beta\partial_t^4\psi + |\psi|^2\psi$$

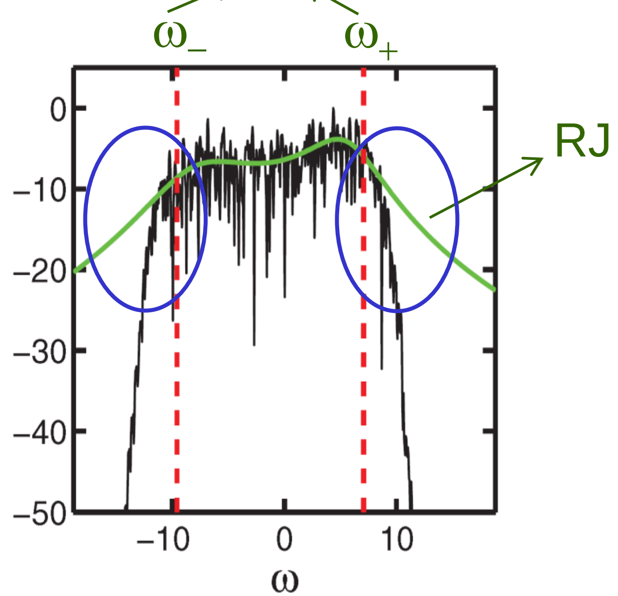
WT →

$$\omega_\pm = -\frac{\alpha}{4\beta} \pm \frac{\sqrt{21}}{12\beta}\sqrt{3\alpha^2 + 8\beta}$$

→ RJ: compactly supported spectral shape

Barviau, Garnier, Xu, Picozzi -- PRA (2013)

‘Regularize the UV catastrophe’



b) 1 ZDW: Anomalous thermalization

Breakdown of thermalization… FPU problem

→ The Fermi-Pasta-Ulam Problem: A Status Report,
Ed. by G. Gallavotti, Lecture Notes in Physics (Springer, New York, 2007)

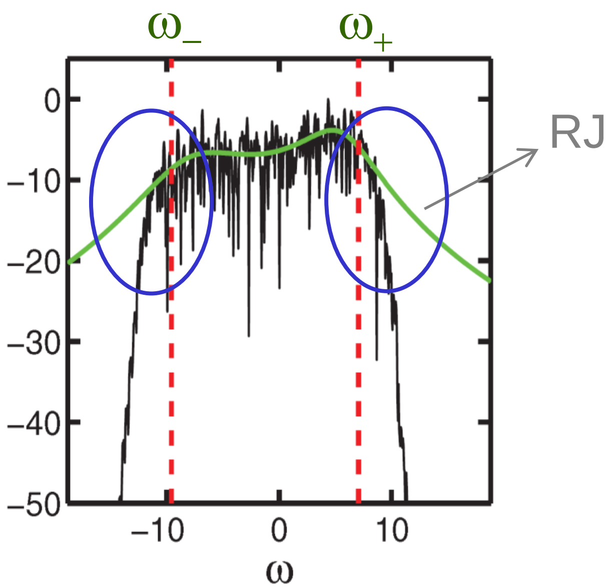
a) 2 ZDWs: Truncated RJ

$$i\partial_z\psi = -\partial_t^2\psi + i\alpha\partial_t^3\psi + \beta\partial_t^4\psi + |\psi|^2\psi$$

$$\omega_\pm = -\frac{\alpha}{4\beta} \pm \frac{\sqrt{21}}{12\beta}\sqrt{3\alpha^2+8\beta}$$

→ RJ: compactly supported spectral shape

Barviau, Garnier, Xu, Picozzi -- PRA (2013)



b) 1 ZDW: Anomalous thermalization

$$i\partial_z\psi = -\partial_t^2\psi + i\alpha\partial_t^3\psi + |\psi|^2\psi$$

$$n^{RJ}(\omega) = \frac{T}{k(\omega)+\lambda\omega-\mu}$$: not even well-defined → $N = T\int n^{RJ}(\omega,z)\,d\omega$ diverge

b) 1 ZDW: Breakdown of thermalization

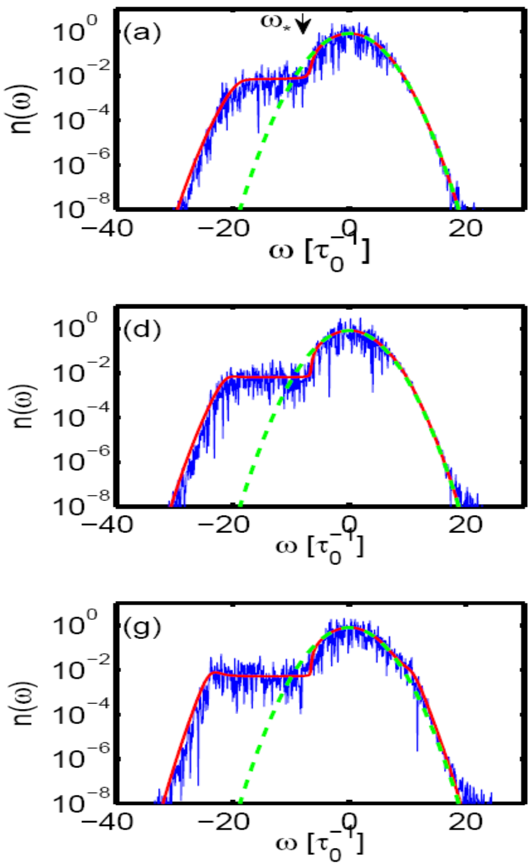
$$i\partial_z A = -\partial_t^2 A - i\alpha\partial_t^3 A + |A|^2 A$$

$$\begin{cases} J_\omega = n(z,\omega) - n(z,q-\omega) \\ q = 2/3\alpha \end{cases}$$

$$\partial_z n(\omega,z) = \frac{1}{3\pi|\alpha|}\int \frac{n_\omega(n_\omega - J_\omega)n_{\omega_1}(n_{\omega_1} - J_{\omega_1})}{|\omega-\omega_1|\,|\omega+\omega_1-q|}\left(\frac{1}{n_\omega} + \frac{1}{n_\omega - J_\omega} - \frac{1}{n_{\omega_1}} - \frac{1}{n_{\omega_1} - J_{\omega_1}}\right) d\omega_1$$

$$\partial_z \mathcal{S} \geq 0 \qquad \mathcal{S}(z)/T_0 = \frac{1}{2\pi}\int \log[n_\omega(z)]\, d\omega \qquad N/T_0 = \frac{1}{2\pi}\int n^{loc}(\omega)\, d\omega$$

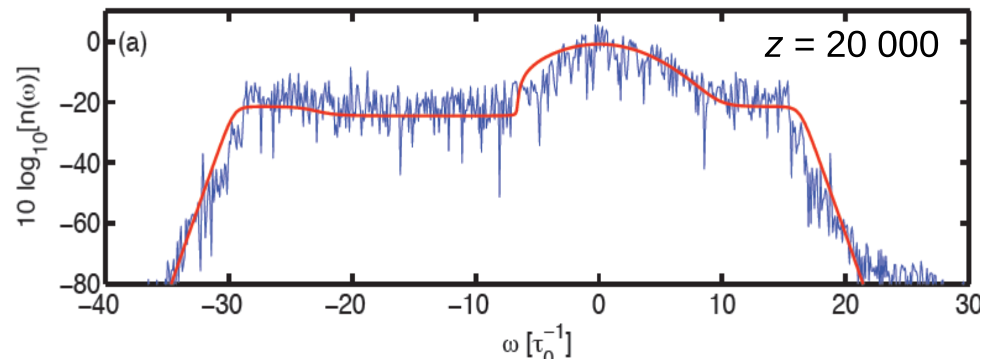
$$n^{loc}(\omega) = J_\omega/2 + [1 + \sqrt{1+\lambda^2 J_\omega^2/4}]/\lambda$$

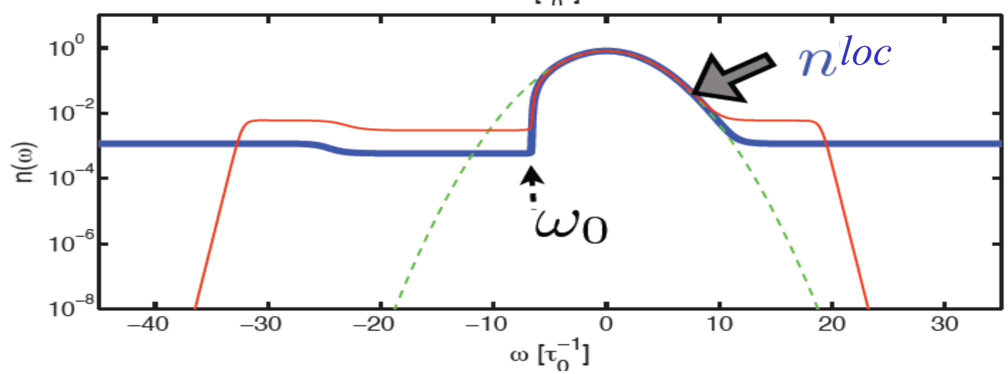


z = 30 L_{nl}

z = 500 L_{nl}

z = 2000 L_{nl}





Suret, Randoux, Jauslin, Picozzi -- PRL (2010)

Michel, Suret, Randoux, Jauslin, Picozzi -- OL (2010)

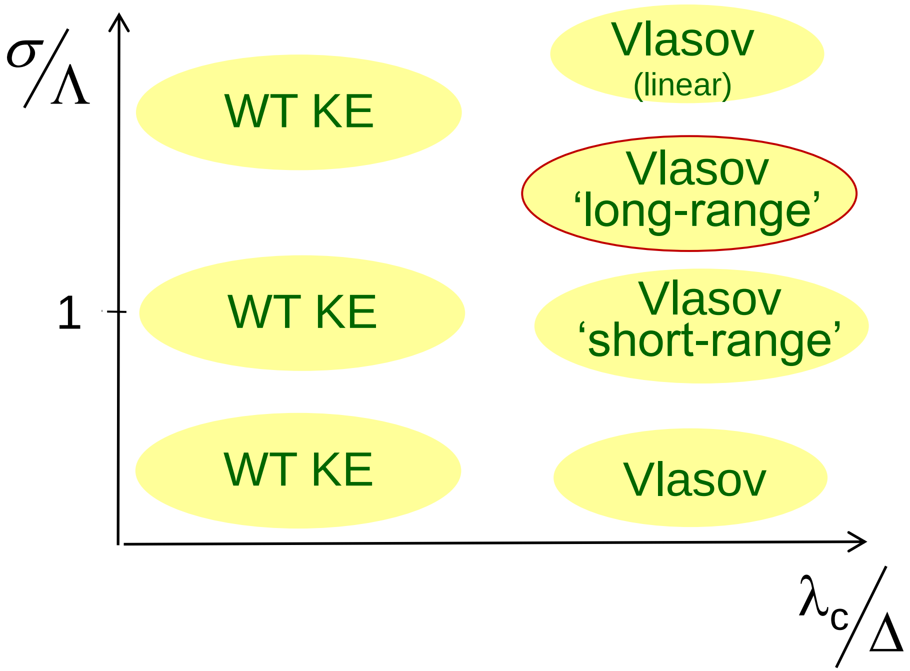
Optical wave turbulence

1.- WT kinetic equation: Thermalization & Condensation:

Wave turbulence in a trap

2.- **Long-range Vlasov Turbulence:**

Breakdown of thermalization

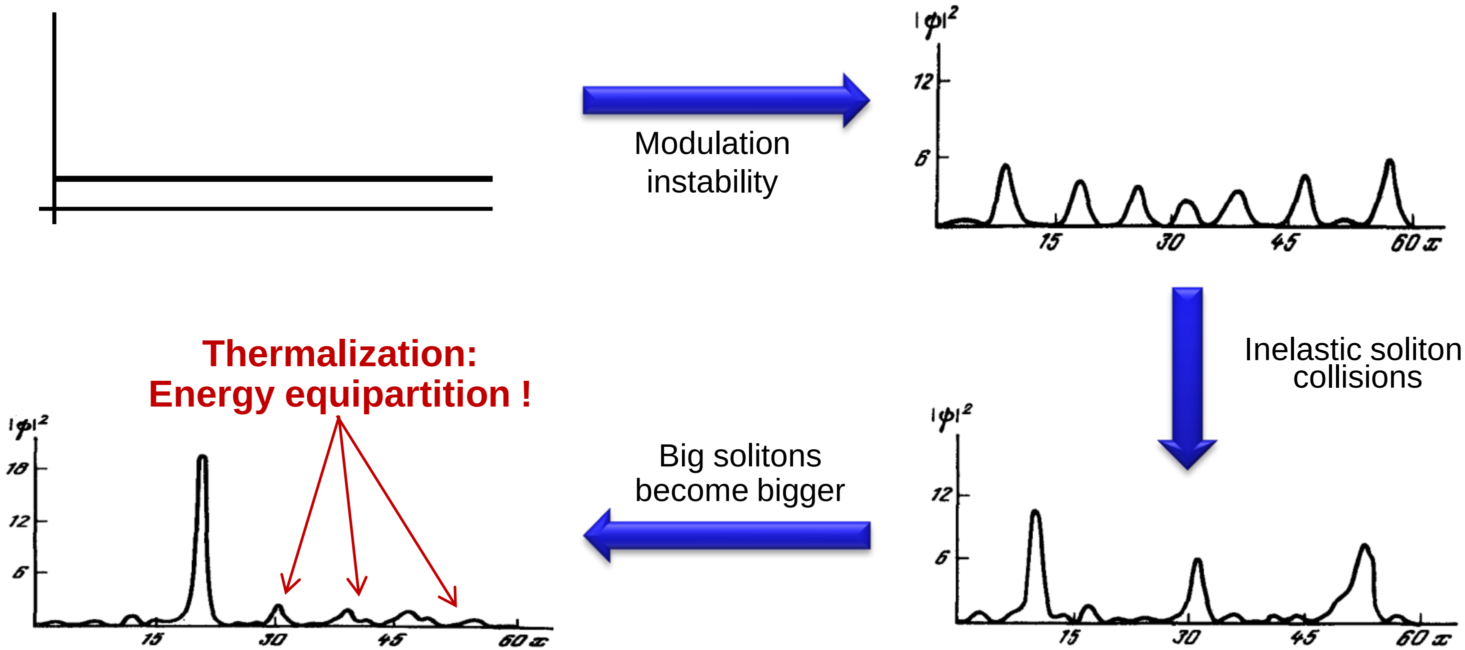


3.- Weak Langmuir Turbulence:

Breakdown of thermalization

‘Soliton turbulence’: Focusing nonlinearity

$$i\partial_z\psi = \partial_{xx}\psi + |\psi|\psi$$ Non-integrable



V.E. Zakharov, et al., Pis’ma Zh. Eksp. Teor. Fiz. 48 (1988) 79 [JETP Lett. 48 (1988) 83]

- B. Rumpf, A.C. Newell, Phys. Rev. Lett. (2001)
- B. Rumpf, A.C. Newell, Physica D (2003) ; PRE (2004)
- B. Rumpf PRE (2004) – PRE (2008) – Physica D (2009)
- K.O. Rasmussen, T. Cretegny, P. Kevrekedis, N. Gronbech-Jensen, PRL (2000)
- M. Johansson, K.O. Rasmussen, PRE (2004); PRE (2006)
- R. Jordan, B. Turkington, C. Zirbel, Physica D (2000)
- R. Jordan, C. Josserand, PRE (2000)
- S. Nazarenko, *Wave Turbulence* (Springer, 2011)

Weak nonlocality ($\sigma \sim \Lambda$) $\longrightarrow$ Soliton turbulence

$$i\partial_z\psi + \beta\partial_{xx}\psi + \gamma\psi \int_{-\infty}^{+\infty} U(x-x')\ |\psi|^2(z,x')\ dx' = 0$$

σ : range of the nonlocal potential

$\Lambda = \sqrt{\beta/(\gamma|\psi|^2)}$: healing length

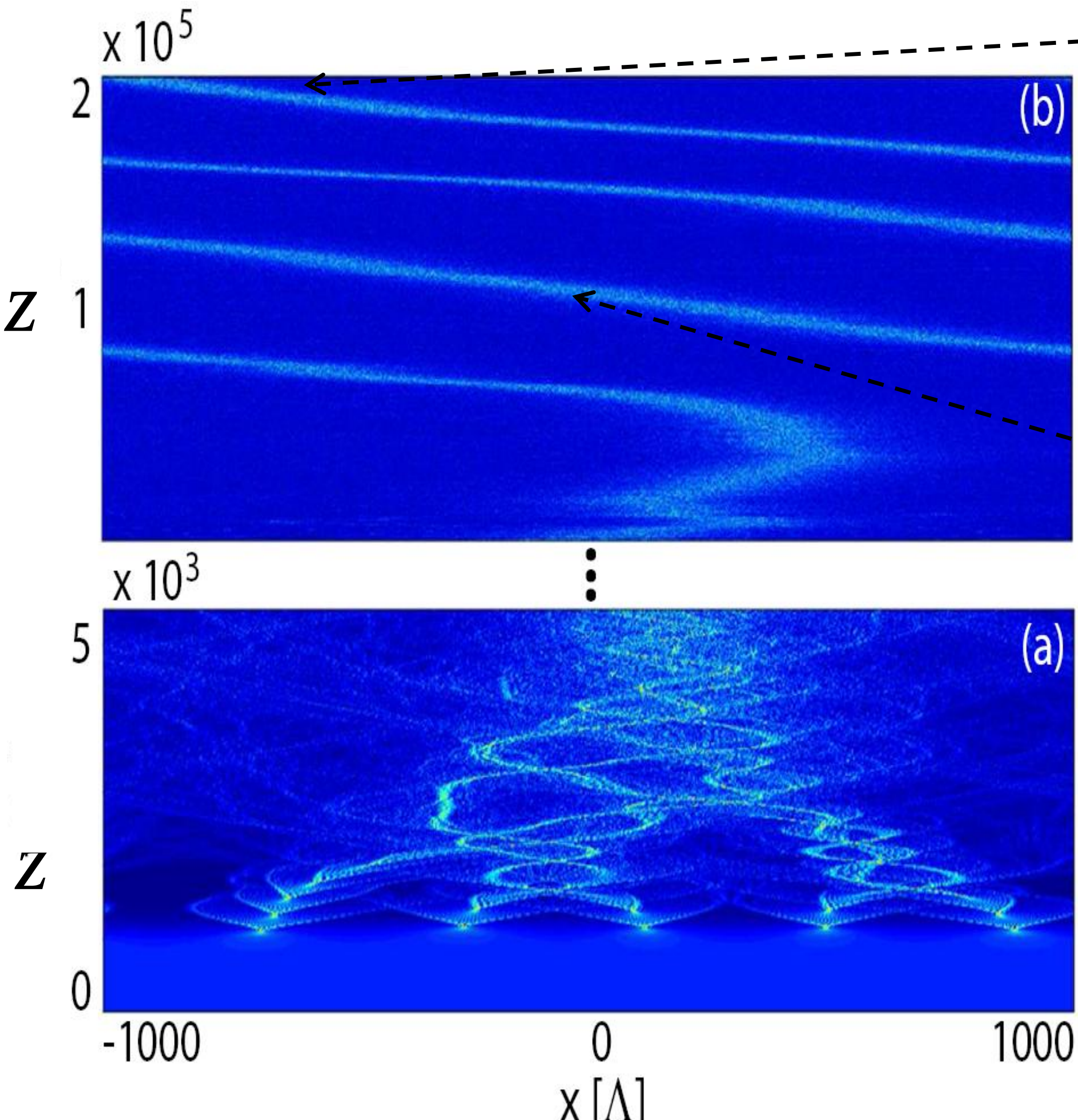
Highly nonlocal nonlinear regime

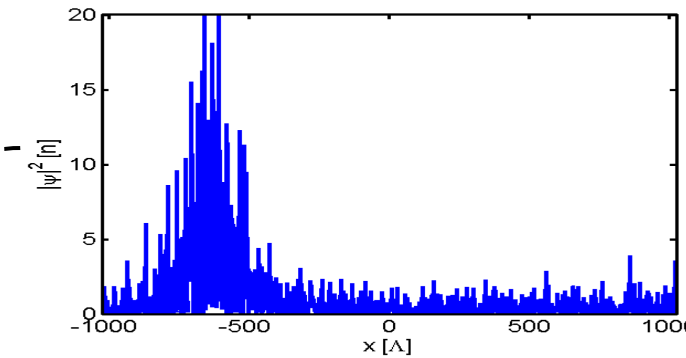
$\left(\Lambda = \sqrt{\beta/\gamma|\psi|^2}\right)$

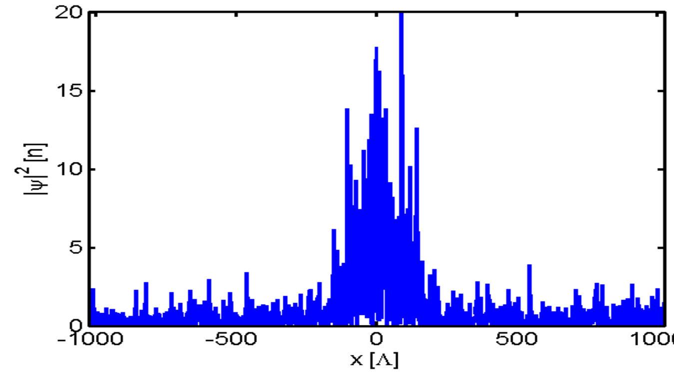
$$i\partial_z\psi + \beta\partial_{xx}\psi + \gamma\psi\int_{-\infty}^{+\infty} U(x-x')\,|\psi|^2(z,x')\,dx' = 0$$

$\sigma \gg \Lambda$

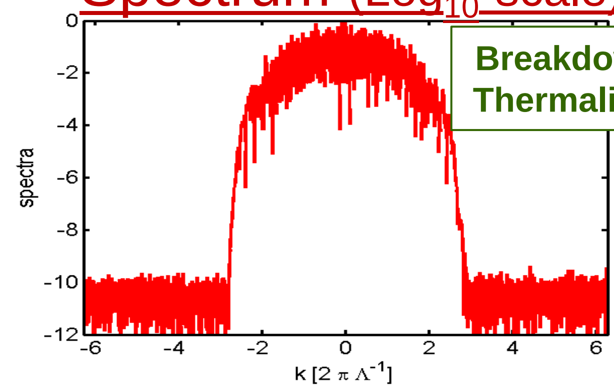
Initial condition: $\psi = 1$







Spectrum (Log_{10}-scale):



Breakdown of Thermalization

Picozzi, Garnier, PRL (2011)

Reminder: Long-range Vlasov equation

$$i\partial_z\psi + \beta\partial_{xx}\psi + \gamma\psi\int_{-\infty}^{+\infty} U(x-x')\ |\psi|^2(z,x')\ dx' = 0$$

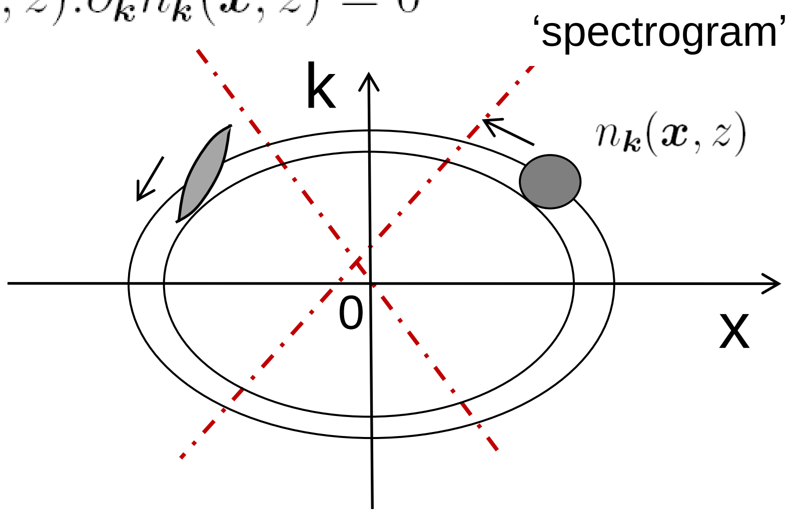
$$B(\boldsymbol{x},\boldsymbol{\xi},z) = \langle\psi(\boldsymbol{x}+\boldsymbol{\xi}/2,z)\psi^*(\boldsymbol{x}-\boldsymbol{\xi}/2,z)\rangle \qquad n_k(\boldsymbol{x},z) = \int B(\boldsymbol{x},\boldsymbol{\xi},z)\exp(-i\boldsymbol{k}\cdot\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}$$

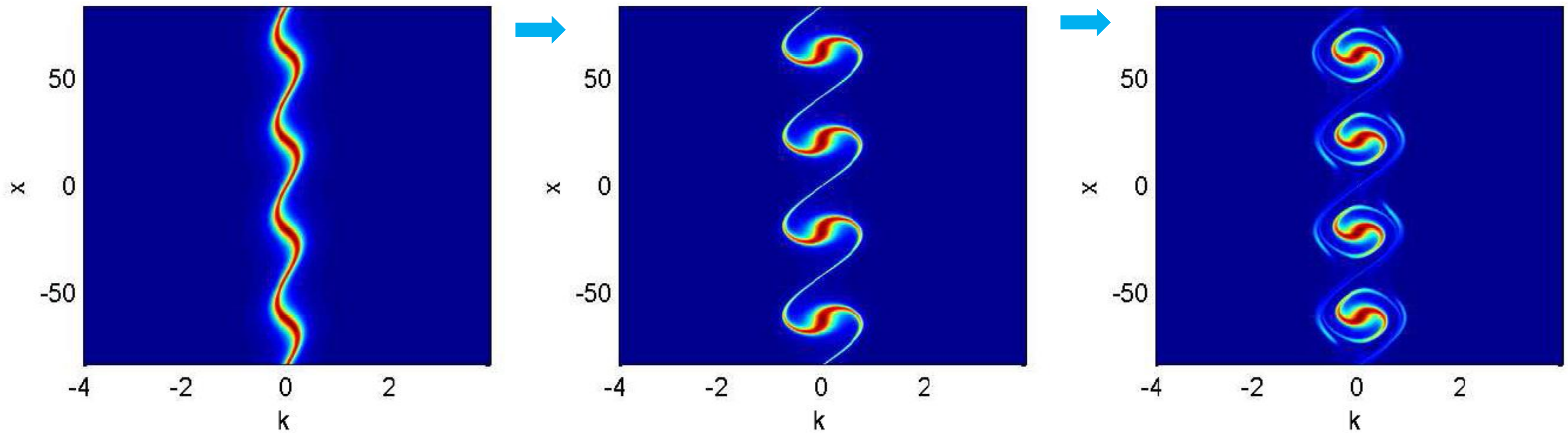
$$\partial_z n_{\boldsymbol{k}}(\boldsymbol{x},z) + \partial_{\boldsymbol{k}}\tilde{\omega}_{\boldsymbol{k}}(\boldsymbol{x},z).\partial_{\boldsymbol{x}} n_{\boldsymbol{k}}(\boldsymbol{x},z) - \partial_{\boldsymbol{x}}\tilde{\omega}_{\boldsymbol{k}}(\boldsymbol{x},z).\partial_{\boldsymbol{k}} n_{\boldsymbol{k}}(\boldsymbol{x},z) = 0$$

$$\begin{cases}\dot{\boldsymbol{x}} = \partial_{\boldsymbol{k}}\tilde{\omega} \\ \dot{\boldsymbol{k}} = -\partial_{\boldsymbol{x}}\tilde{\omega}\end{cases} \qquad \begin{aligned}\tilde{\omega}_{\boldsymbol{k}}(\boldsymbol{x},z) &= \omega(\boldsymbol{k}) + V(\boldsymbol{x},z) \\ \omega(\boldsymbol{k}) &= \beta_s k^2\end{aligned}$$

$$V(\boldsymbol{x},z) = \gamma\int U(\boldsymbol{x}-\boldsymbol{x}')\,N(\boldsymbol{x}',z)\,d\boldsymbol{x}'$$

$$N(\boldsymbol{x},z) = \frac{1}{(2\pi)^d}\int n_{\boldsymbol{k}}(\boldsymbol{x},z)\,d\boldsymbol{k}$$





Particles self-trap into a spiralling behavior ('phase-mixing')

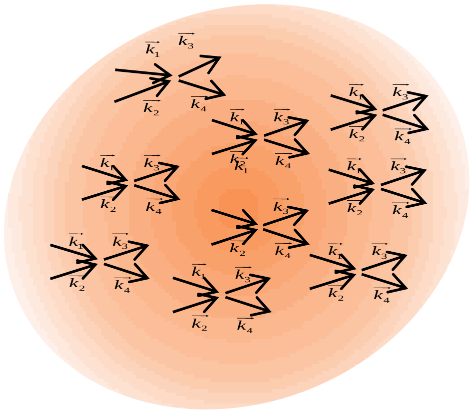
Reminder: Vlasov eqn

$$\partial_z n_{\boldsymbol{k}}(\boldsymbol{x},z) + \partial_{\boldsymbol{k}}\tilde{\omega}_{\boldsymbol{k}}(\boldsymbol{x},z).\partial_{\boldsymbol{x}} n_{\boldsymbol{k}}(\boldsymbol{x},z) - \partial_{\boldsymbol{x}}\tilde{\omega}_{\boldsymbol{k}}(\boldsymbol{x},z).\partial_{\boldsymbol{k}} n_{\boldsymbol{k}}(\boldsymbol{x},z) = 0$$

$$\begin{cases} \dot{\boldsymbol{x}} = \partial_{\boldsymbol{k}}\tilde{\omega} \\ \dot{\boldsymbol{k}} = -\partial_{\boldsymbol{x}}\tilde{\omega} \end{cases} \qquad \begin{aligned} \tilde{\omega}_{\boldsymbol{k}}(\boldsymbol{x},z) &= \omega(\boldsymbol{k}) + V(\boldsymbol{x},z) \\ \omega(\boldsymbol{k}) &= \beta_s k^2 \end{aligned}$$

$$V(\boldsymbol{x},z) = \gamma \int U(\boldsymbol{x}-\boldsymbol{x}')\, N(\boldsymbol{x}',z)\, d\boldsymbol{x}'$$

$$N(\boldsymbol{x},z) = \frac{1}{(2\pi)^d}\int n_{\boldsymbol{k}}(\boldsymbol{x},z)\, d\boldsymbol{k}$$



Breakdown of thermalization
(analogy with gravitation)

- $\mathcal{M} = \iint f[n]\, d\boldsymbol{x} d\boldsymbol{k}$

  $f[n]$ is an arbitrary functional of $n_{\boldsymbol{k}}(\boldsymbol{x},z)$ $\quad \big(f[n] = \log(n)\big)$

  $$\mathcal{H}_{Vl} = \iint \omega(\boldsymbol{k})\, n_{\boldsymbol{k}}(\boldsymbol{x},z)\, d\boldsymbol{x} d\boldsymbol{k} + \frac{1}{2}\int V(\boldsymbol{x}) N(\boldsymbol{x})\, d\boldsymbol{x}$$

- Formal reversibility $(z,\boldsymbol{k}) \to (-z,-\boldsymbol{k})$

Soliton solution of the 'long-range' Vlasov eqn

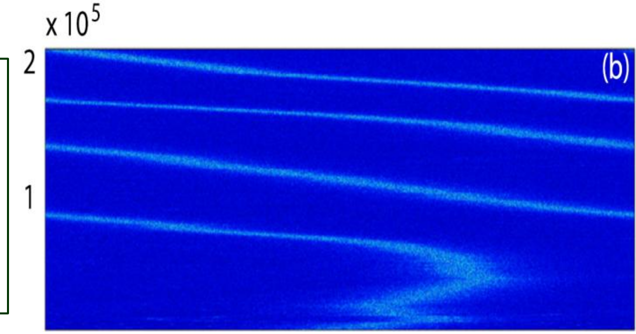
$$\begin{cases} N(x) = \mathrm{n}(2\pi\sigma_N^2)^{-1/2}\exp[-x^2/(2\sigma_N^2)] \\ U(x) = (2\pi\sigma^2)^{-1/2}\exp(-x^2/(2\sigma^2)) \end{cases}$$

Local limit: $U(x) = \delta(x)$
Hasegawa (1975)

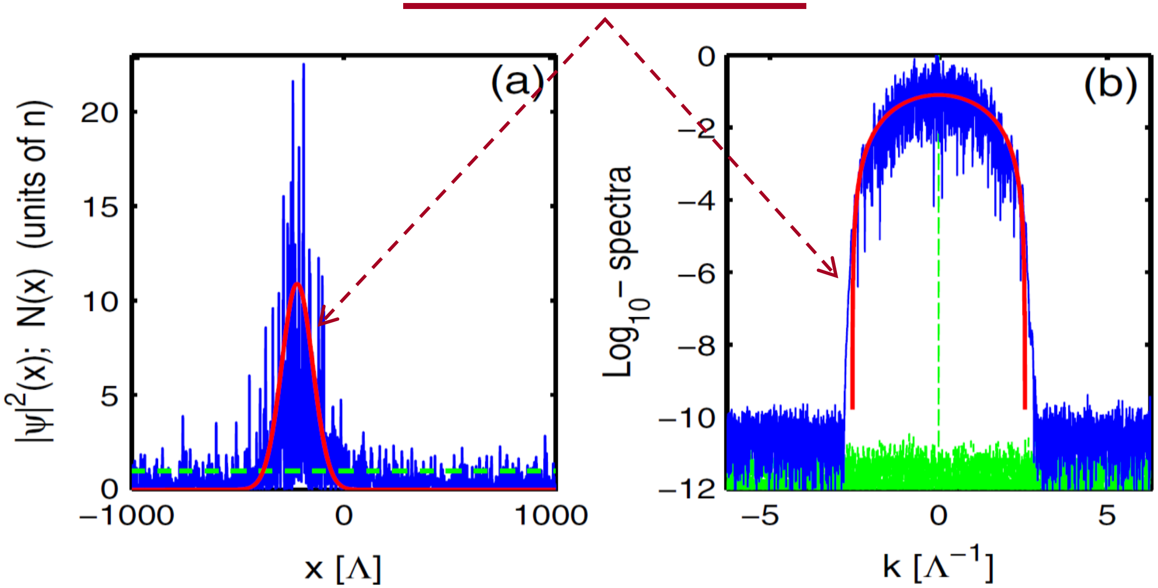
$$n_k^{st}(x+vz) = Q_\alpha\left[c_\alpha\gamma N^\alpha(x+vz) - \beta(k+v/(2\beta))^2\right]^{\frac{1}{\alpha}-\frac{1}{2}}$$

$$Q_\alpha = 2\pi\beta^{\frac{1}{2}}\Gamma(\alpha^{-1}+1)/[\Gamma(\alpha^{-1}+1/2)\Gamma(1/2)(c_\alpha\gamma)^{1/\alpha}]$$

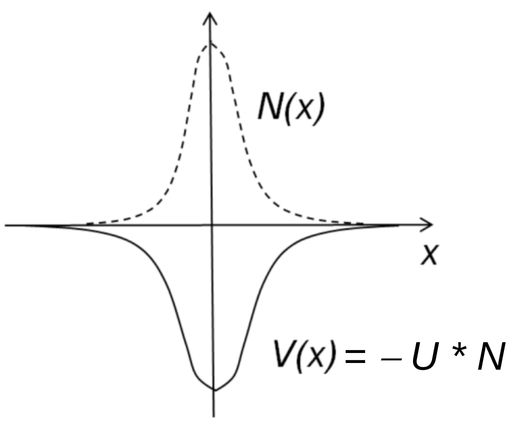
$$\alpha = (1+(\sigma/\sigma_N)^2)^{-1} \qquad c_\alpha = (2\pi)^{\frac{\alpha}{2}-\frac{1}{2}}\sigma_N^\alpha \mathrm{n}^{1-\alpha}(\sigma^2+\sigma_N^2)^{-\frac{1}{2}}$$



Soliton solution

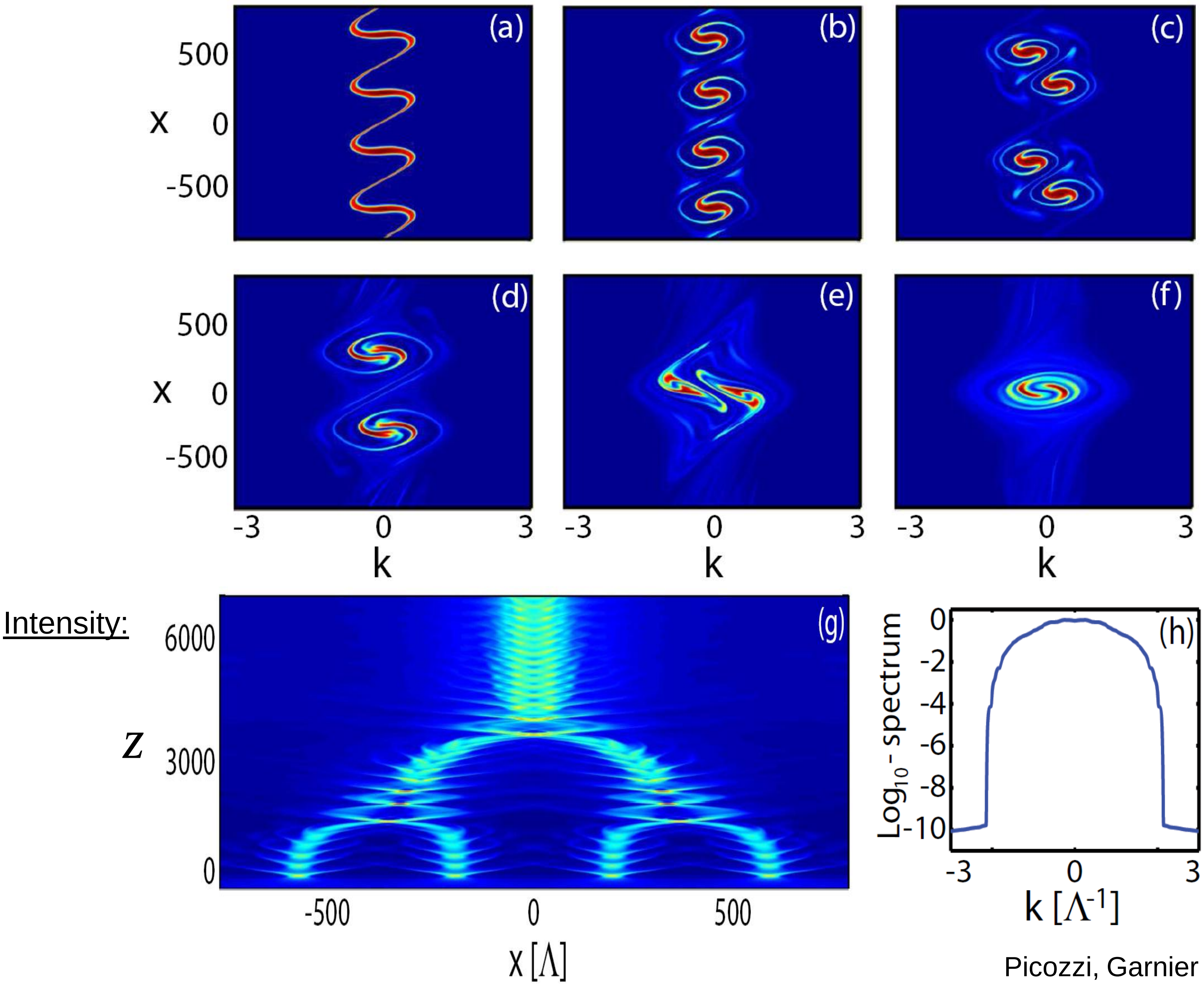


Trapping condition: $k < k_c$



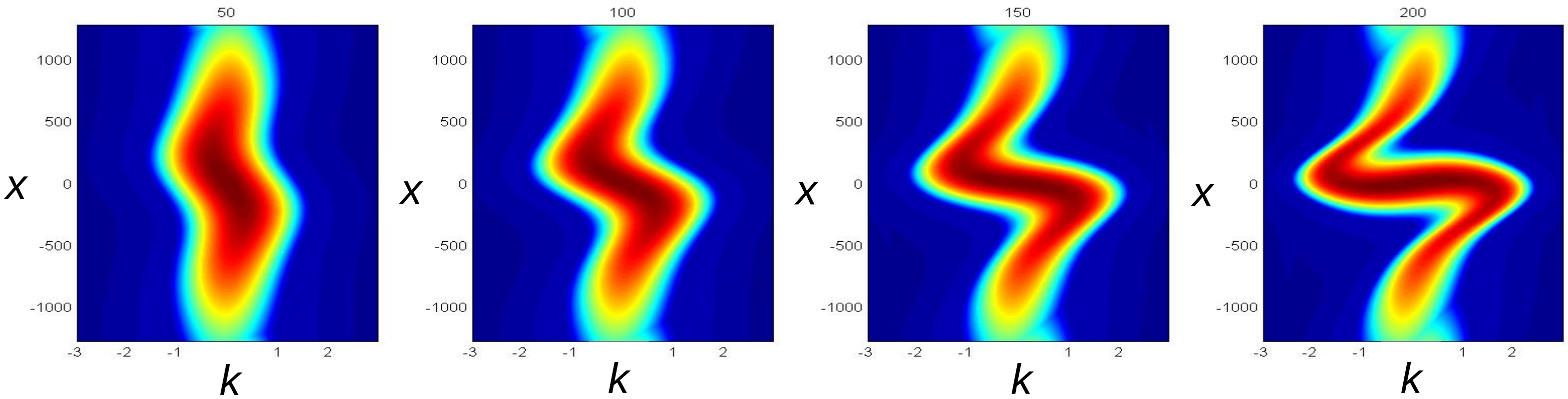
$$\boxed{\beta k^2 - V(x) \leq 0} \qquad k_c \sim \sqrt{V/\beta} \qquad k_c^{-1} \sim \Lambda\sqrt{\sigma/L}$$

Simulation ‘long-range’ Vlasov eqn.



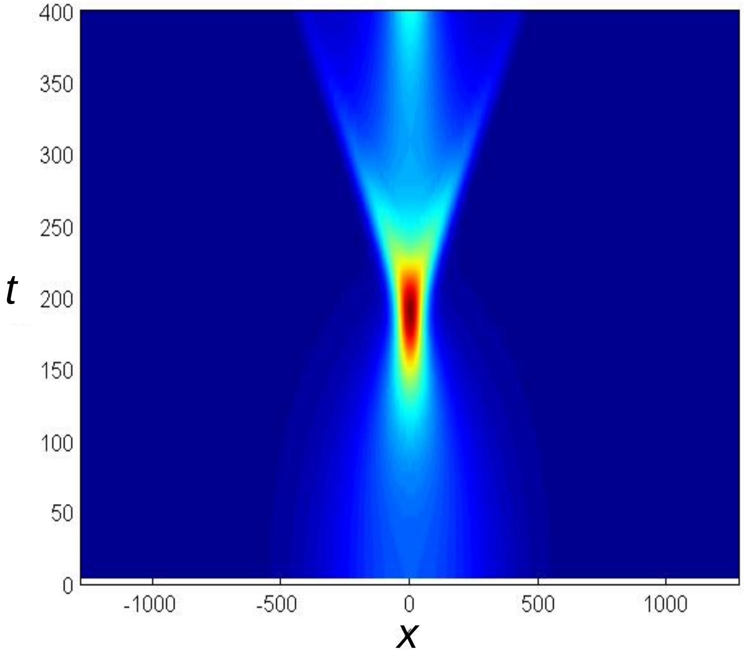
Picozzi, Garnier -- PRL (2011)

Singular behaviors of incoherent waves



Log$_{10}$-scale

Intensity: N(x)



In collaboration with:
D. Faccio, D. Vocke, T. Roger
S. Trillo

Incoherent rogue waves

Singular solutions Vlasov eqn

⟶ Incoherent shocks & Blow-up singularities

$$\begin{cases} \partial_t K + \beta K \partial_x K + \partial_x V = 0 \\ \partial_t N + \beta \partial_x (NK) = 0 \end{cases}$$

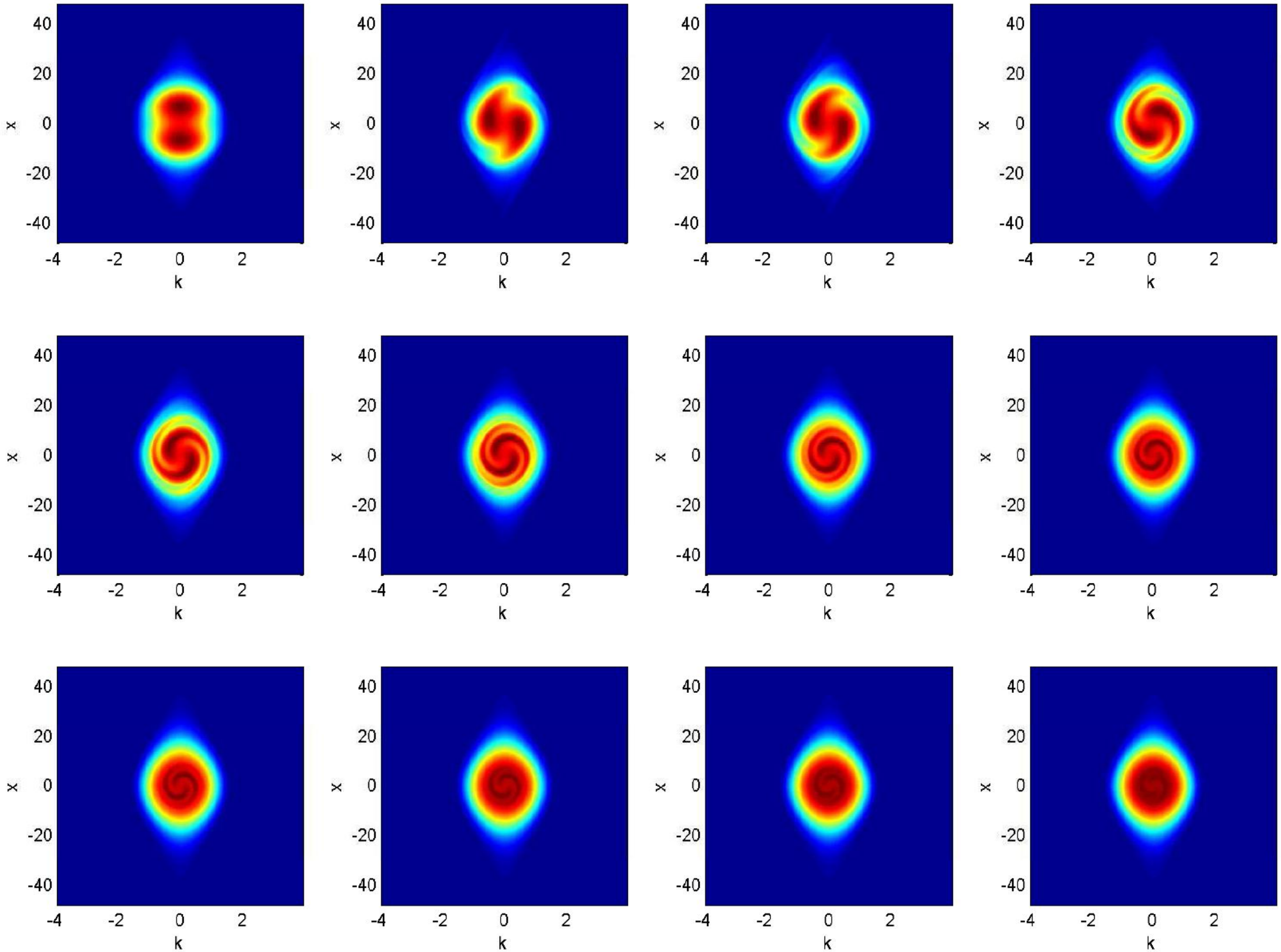
$$\begin{cases} \xi(t) = \partial_x K(t, X(t)) \\ \phi(t) = N(t, X(t)) \end{cases}$$

$$\begin{cases} \dfrac{d\xi}{dt} = -\partial_x^2 V(t, X(t)) - \beta \xi^2(t) \\ \dfrac{d\phi}{dt} = -\beta \xi(t) \phi(t) \end{cases}$$

$$\partial_x K(t_\infty - t, 0) \simeq -\frac{1}{\beta t} \qquad N(t_\infty - t, 0) \simeq \frac{N_0(0)}{\sqrt{\beta}\xi_0 t}$$

Nonlinear Landau damping

⟶ Nonlinear soliton stability



C. Mouhot and C. Villani *Acta Mathematica* 207 (september 2011), 29-201.

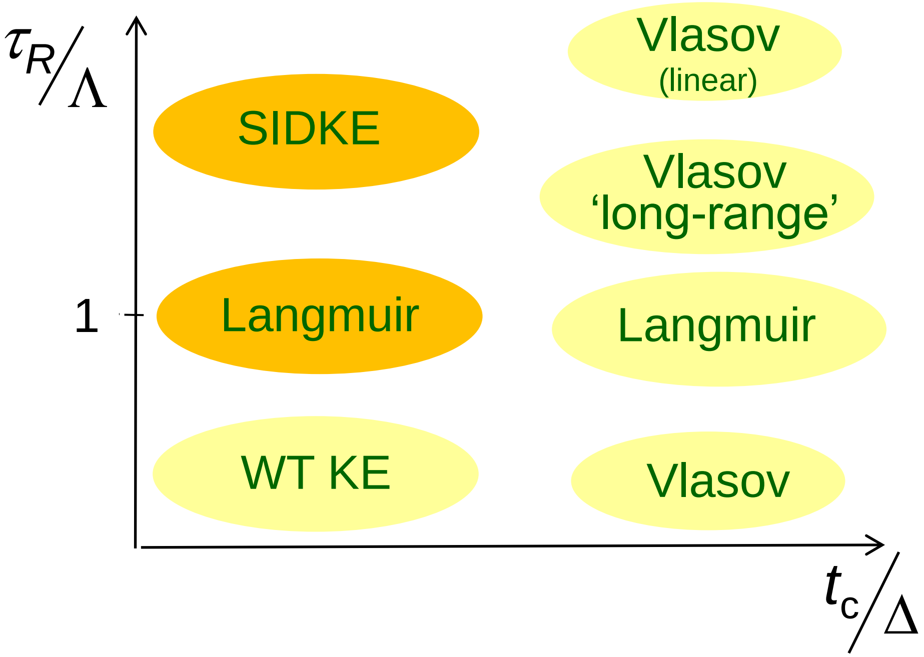
Optical wave turbulence

1.- WT kinetic equation: Thermalization & Condensation:

Wave turbulence in a trap

2.- Long-range Vlasov Turbulence:

3.- Weak Langmuir Turbulence:



Reminder: Spectral incoherent solitons

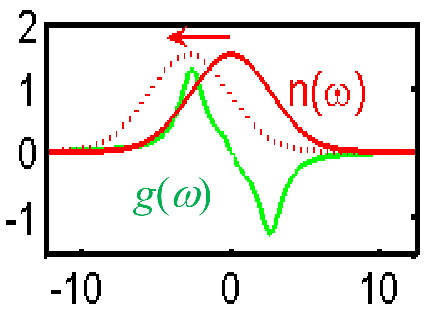
$$i\partial_z \psi = -\beta_t \partial_{tt}\psi + \gamma\psi \int_{-\infty}^{+\infty} R(t-t')\ |\psi|^2(z,t')\ dt'$$

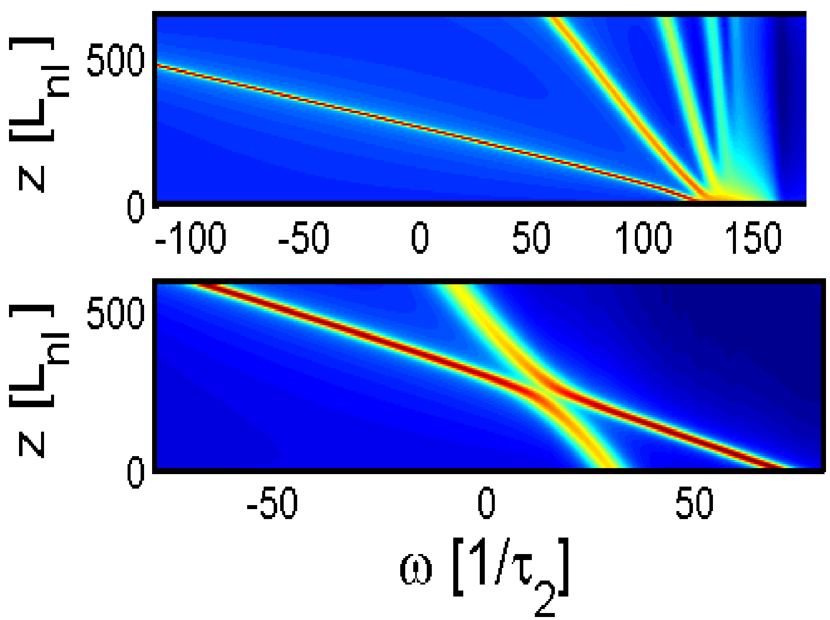
$$R(t) = 0 \text{ for } t < 0$$
$$\tilde{R}(\omega) = \tilde{U}(\omega) + i\,g(\omega)$$

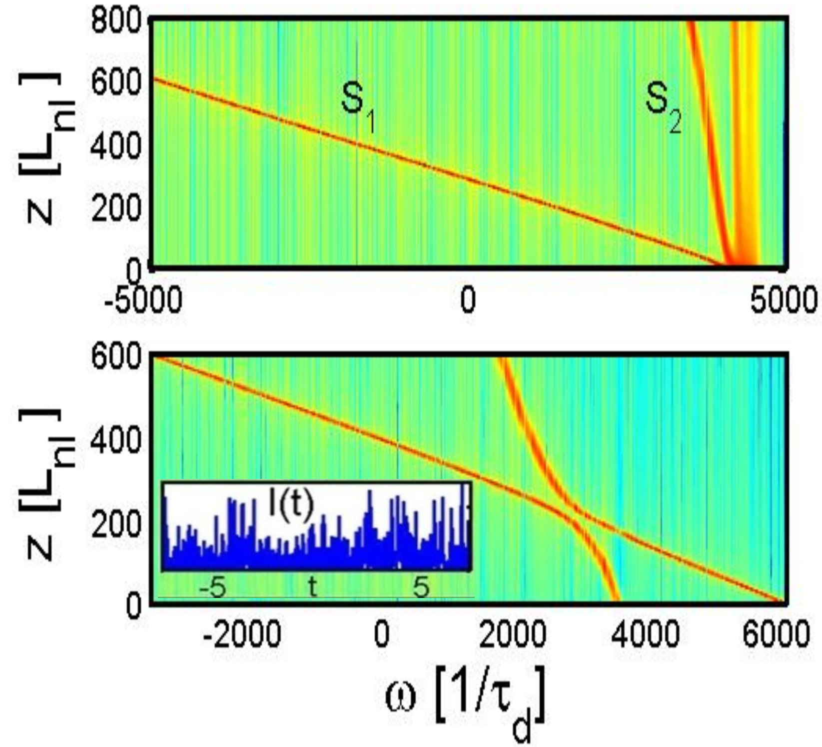
↓ WT

$$\partial_z n_\omega(z) = \frac{\gamma}{\pi}\, n_\omega(z) \int_{-\infty}^{+\infty} g(\omega-\omega')\, n_{\omega'}(z)\, d\omega'$$

$$\partial_z S = 0\,!$$



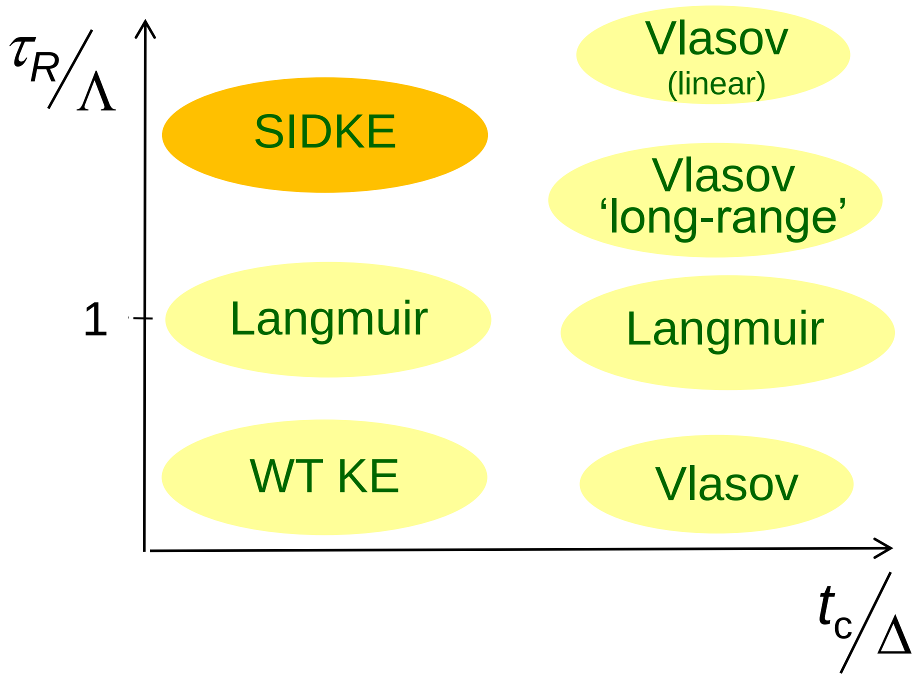




Picozzi, Pitois, Millot PRL (2008)
Michel, Kibler, Picozzi PRA (2011)
Kibler, Michel, Barviau, Kudlinski, Millot, Picozzi, PRE (2011)
Zakharov, Muscher, Rubenchik, Phys. Rep. (1995)

Long-range interaction: Temporal domain

⟶ Singular integro-differential kinetic equations

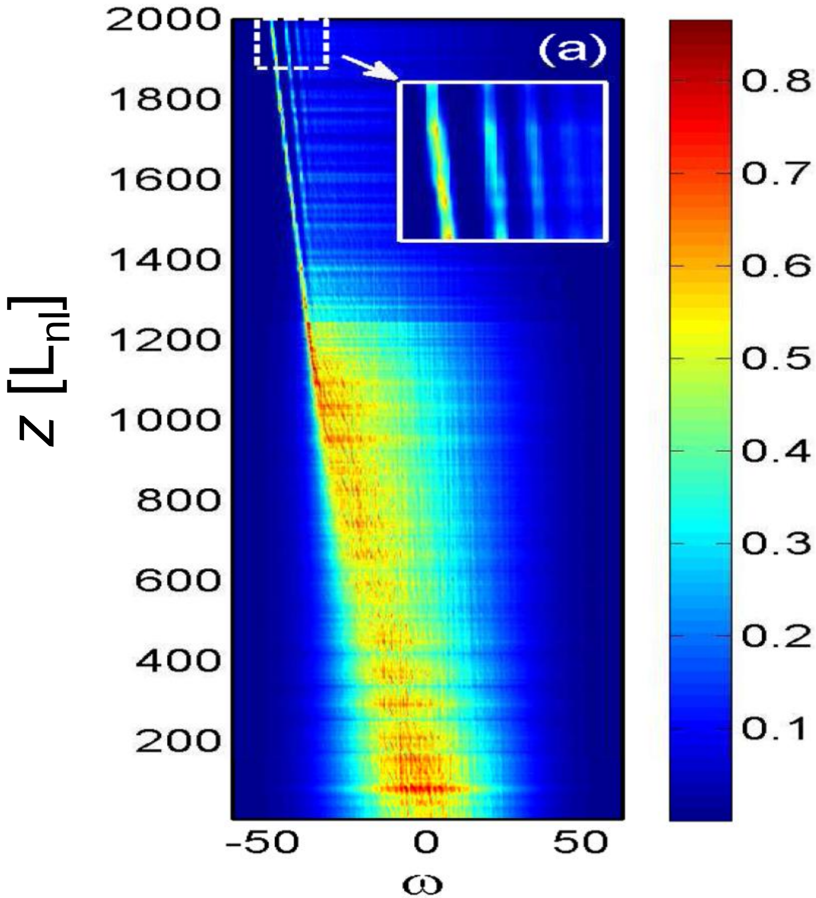


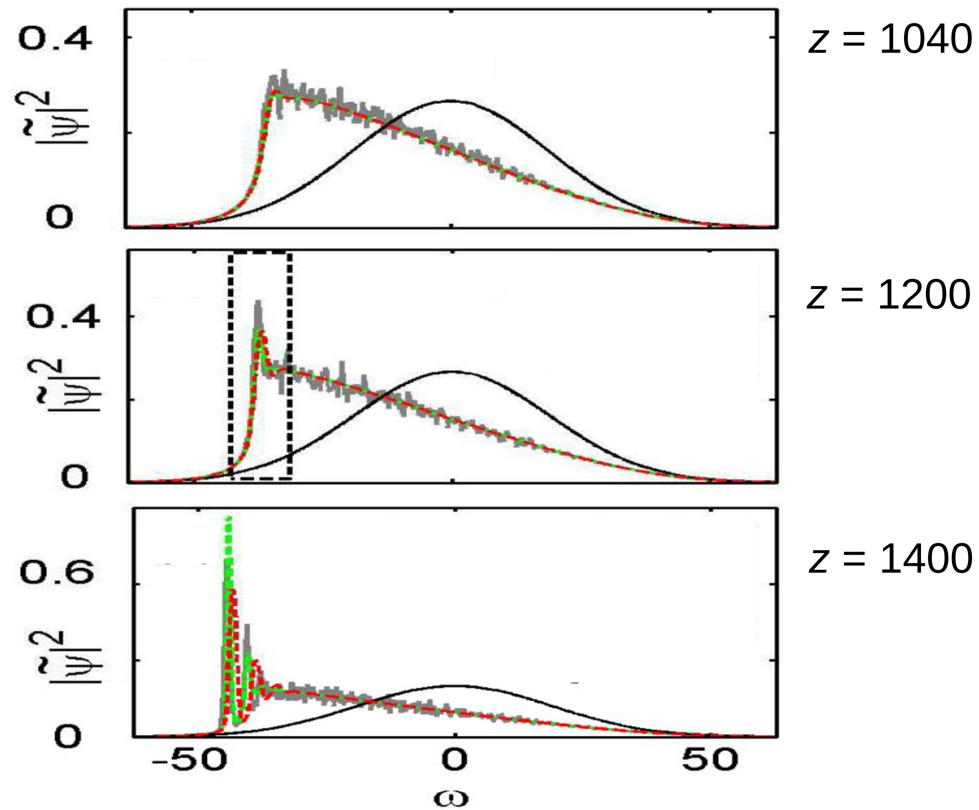
In collaboration with Stefano Trillo

Incoherent dispersive shock waves
in the spectral evolution of random waves

$$i\partial_z\psi = -\beta_t\partial_{tt}\psi + \gamma\psi\int_{-\infty}^{+\infty} R(t-t')\ |\psi|^2(z,t')\ dt'$$

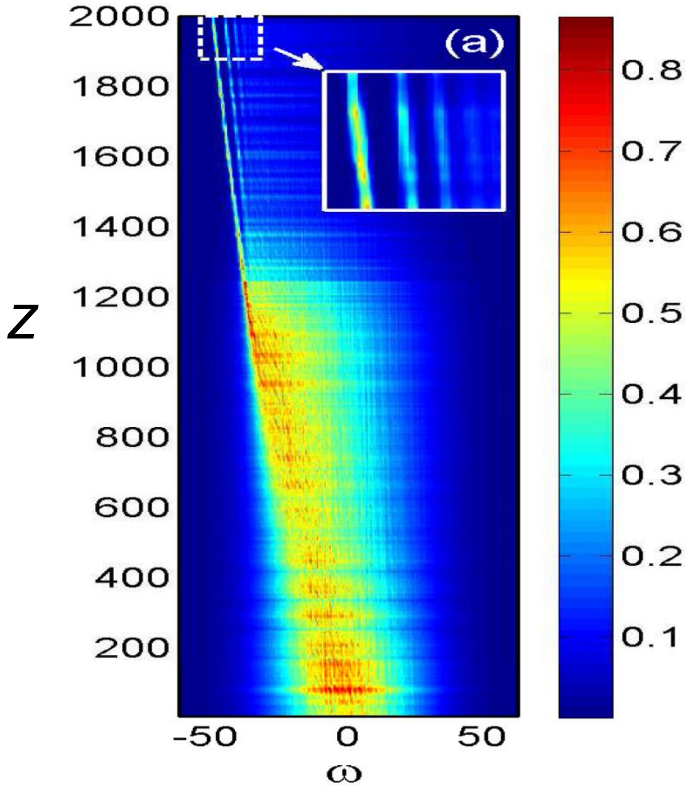
Raman response function: $R(t) = \frac{1+\beta^2}{\beta\tau_R}\sin(\beta t/\tau_R)\exp(-t/\tau_R)H(t)$

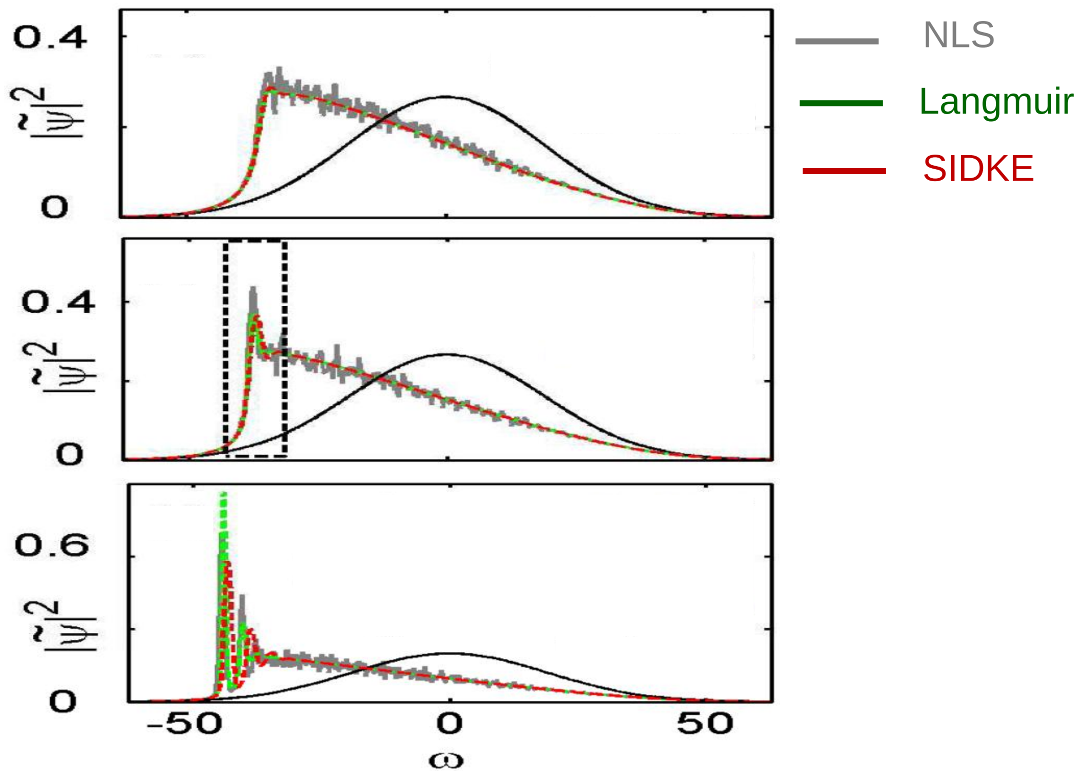




Garnier, Xu, Trillo, Picozzi -- PRL (2013)
Xu, Garnier, Trillo, Picozzi -- PRA (2014)

Incoherent dispersive shock waves





Damped harmonic oscillator:

$$R(t) = \frac{1+\beta^2}{\beta\tau_R}\sin(\beta t/\tau_R)\exp(-t/\tau_R)H(t)$$

$$\partial_z n_\omega = \frac{1}{\pi} n_\omega \int_{-\infty}^{+\infty} g_{\omega-\omega'} n_{\omega'} d\omega'$$

Singular integro-differential kinetic eqn:

$$\tau_R^2 \partial_z n_\omega = (1+\beta^2)\Big(n_\omega \partial_\omega n_\omega - \frac{1}{\tau_R} n_\omega \mathcal{H}\partial_\omega^2 n_\omega\Big)$$

$$\mathcal{H}f(\omega) = \pi^{-1}\mathcal{P}\int_{-\infty}^{+\infty}\frac{f(\omega-u)}{u}du$$

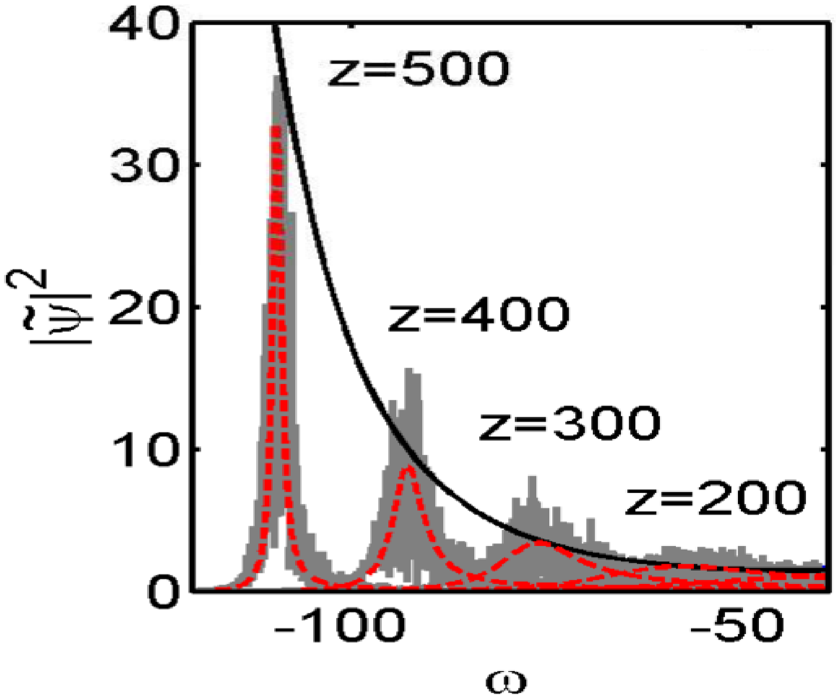
Garnier, Xu, Trillo, Picozzi -- PRL (2013)
Xu, Garnier, Trillo, Picozzi -- PRA (2014)

Inhibition of DSWs: Spectral Collapse Singularity

Exponential response: $R(t) = H(t)\frac{1}{\tau_R}\exp\left(-\frac{t}{\tau_R}\right)$

$$\tau_R \partial_z n_\omega = -n_\omega \mathcal{H} n_\omega - \frac{1}{\tau_R} n_\omega \partial_\omega n_\omega + \frac{1}{2\tau_R^2} n_\omega \mathcal{H} \partial_\omega^2 n_\omega$$

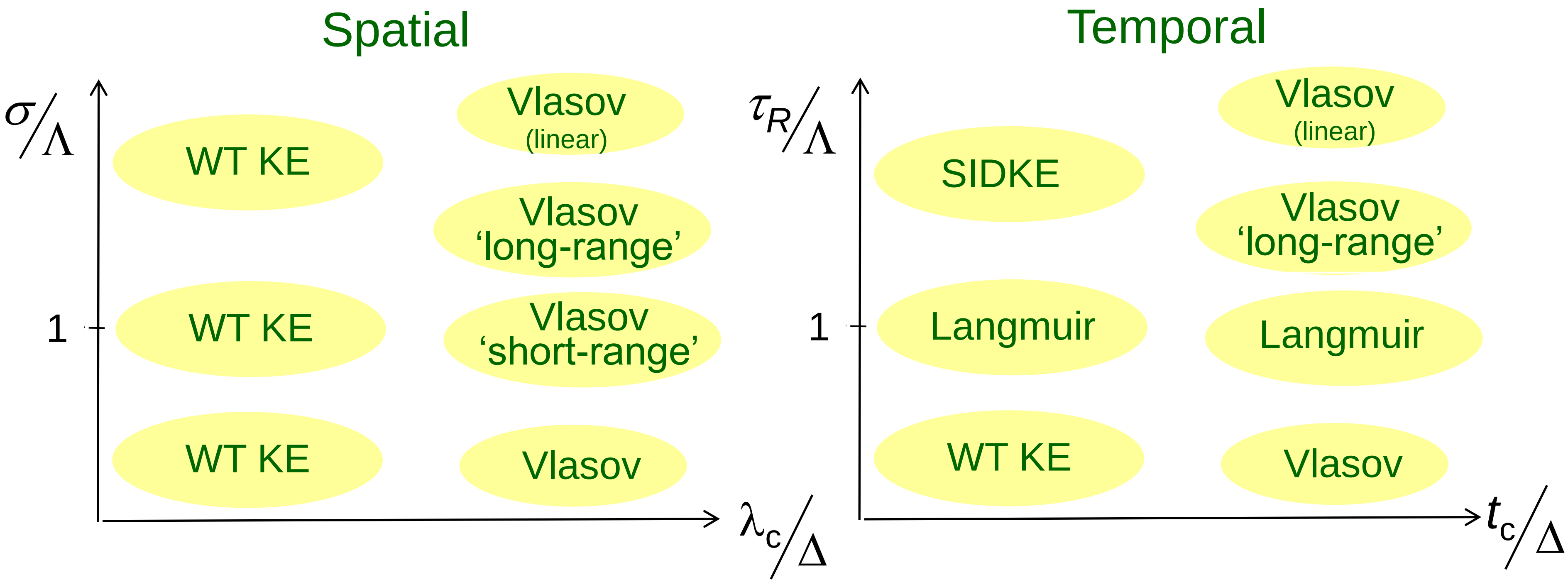
$$n_\omega(\tilde{z}) = \frac{4n_\omega^0}{(2+\tilde{z}\mathcal{H}n_\omega^0)^2 + \tilde{z}^2(n_\omega^0)^2}$$ → Collapse



→ **Coherence enhancement by 2 order of magnitudes**

Garnier, Xu, Trillo, Picozzi -- PRL (2013)
Xu, Garnier, Trillo, Picozzi -- PRA (2014)

Optical wave turbulence



<u>Physics Reports</u>:

Optical Wave Turbulence

Toward a unified nonequilibrium thermodynamic formulation of statistical nonlinear optics

Picozzi, Garnier, Hansson, Suret, Randoux, Millot, Christodoulides

Physics Reports 542, 1-132 (2014)

<u>Works in progress (*submitted…*):</u>

- Vlasov formalism: **Incoherent singularities** (shocks, collapse…)
- WT formalism: **Violation of *H*-theorem of entropy growth**

  → Emergence of mutual coherence : Reduction of nonequilibrium entropy

Extension to disordered systems…?

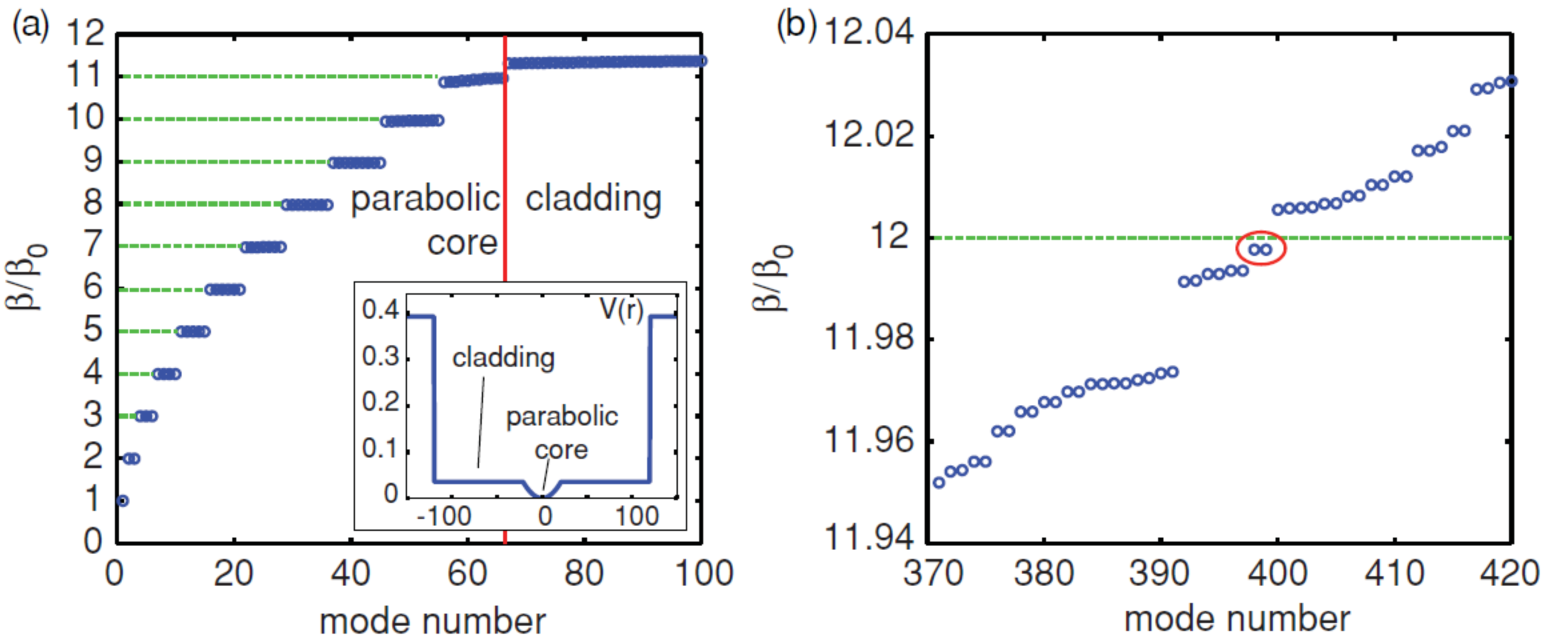
$$i\partial_t\psi = -\nabla^2\psi + V(\boldsymbol{r})\psi + |\psi|^2\psi \qquad \psi(\boldsymbol{r},t) = \sum_m c_m(t)\, u_m(\boldsymbol{r})\, \exp(-i\omega_m t)$$

$$\begin{cases} \partial_t n_{\boldsymbol{\kappa}}(t) = \dfrac{4\pi\gamma^2}{\omega_0^6} \displaystyle\iiint d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2 d\boldsymbol{\kappa}_3 \delta(\omega_{\boldsymbol{\kappa}_1} + \omega_{\boldsymbol{\kappa}_3} - \omega_{\boldsymbol{\kappa}_2} - \omega_{\boldsymbol{\kappa}}) \\ \qquad \times |\tilde{W}_{\boldsymbol{\kappa}\boldsymbol{\kappa}_1\boldsymbol{\kappa}_2\boldsymbol{\kappa}_3}|^2 n_{\boldsymbol{\kappa}} n_{\boldsymbol{\kappa}_1} n_{\boldsymbol{\kappa}_2} n_{\boldsymbol{\kappa}_3} \left(n_{\boldsymbol{\kappa}}^{-1} + n_{\boldsymbol{\kappa}_2}^{-1} - n_{\boldsymbol{\kappa}_1}^{-1} - n_{\boldsymbol{\kappa}_3}^{-1}\right) \\ \qquad + \dfrac{8\pi}{\omega_0^2} \displaystyle\int d\boldsymbol{\kappa}_1 \delta(\omega_{\boldsymbol{\kappa}_1} - \omega_{\boldsymbol{\kappa}}) |\tilde{U}_{\boldsymbol{\kappa}\boldsymbol{\kappa}_1}(\mathbf{n})|^2 (n_{\boldsymbol{\kappa}_1} - n_{\boldsymbol{\kappa}}) \\ \tilde{W}_{\boldsymbol{\kappa}\boldsymbol{\kappa}_1\boldsymbol{\kappa}_2\boldsymbol{\kappa}_3} = \displaystyle\int u_{\boldsymbol{\kappa}}^*(\boldsymbol{r}) u_{\boldsymbol{\kappa}_1}(\boldsymbol{r}) u_{\boldsymbol{\kappa}_2}^*(\boldsymbol{r}) u_{\boldsymbol{\kappa}_3}(\boldsymbol{r})\, d\boldsymbol{r} \\ \tilde{U}_{\boldsymbol{\kappa}\boldsymbol{\kappa}_1}(n) = \dfrac{1}{\omega_0^2} \displaystyle\int d\boldsymbol{\kappa}'\, \tilde{W}_{\boldsymbol{\kappa}\boldsymbol{\kappa}_1\boldsymbol{\kappa}'\boldsymbol{\kappa}'}\, n_{\boldsymbol{\kappa}'} \end{cases}$$

→ Disorder prevents thermalization ?
Thermalization and condensation on Anderson modes ?

→ **C. Michel (Un. Nice) ; J. Garnier (ENS-Paris) ; C. Conti (Un. Rome)**

Physical origin of the frequency cutoff



$$\beta_{m_{p1}} + \beta_{m_{p2}} = \beta_{m_{p3}} + \beta^{\mathrm{cl}}$$

$$\mathcal{W}^{\mathrm{cl}} = \frac{1}{\mathcal{N}^{\mathrm{cl}}} \sum_{\{m_{10}\}} \sum_{\{m_{11}\}} \sum_{\{m_{11}\}} \sum_{\{m^{\mathrm{cl}}\}} \left| W_{m_{10},m_{11},m_{11},m^{\mathrm{cl}}} \right|^2$$

$$\mathcal{W}^{\mathrm{co}} = \frac{1}{\mathcal{N}_{11}} \sum_{\{m_9\}} \sum_{\{m_{10}\}} \sum_{\{m_{10}\}} \sum_{\{m_{11}\}} \left| W_{m_9,m_{10},m_{10},m_{11}} \right|^2$$

$$\boxed{\frac{\mathcal{W}^{\mathrm{cl}}}{\mathcal{W}^{\mathrm{co}}} = \frac{6.4 \times 10^{-11}}{1.9 \times 10^{-5}} \sim 10^{-6}}$$