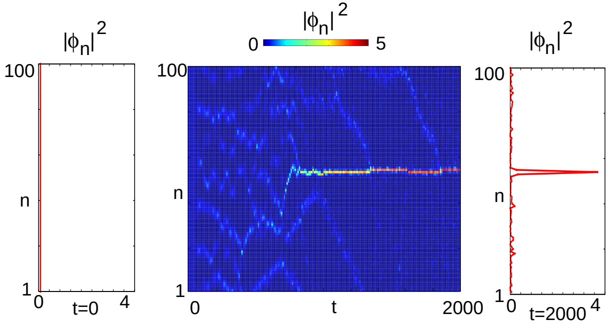
Statistical mechanics of the DNLS: Lessons for turbulent nonequlibrium dynamics

Benno Rumpf,
Mathematics Departments,
Southern Methodist University, Dallas, USA

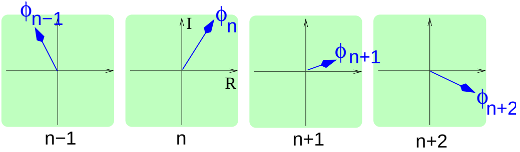


Formation of high-amplitude localised structures
from a sea of random waves

The discrete nonlinear Schrödinger equation

$$i\frac{\partial \phi_n}{\partial t} + \phi_{n+1} + \phi_{n-1} + |\phi_n|^2\phi_n = 0$$

complexe time-dependent field $\phi_n(t)$ at the lattice-site n



coupling-contribution $\phi_{n+1} + \phi_{n-1}$, nonlinearity $|\phi_n|^2\phi_n$

Conserved quatities of the discrete nonlinear Schrödinger-equation

1. norm or 'particle number'

$$A = \sum_n |\phi_n|^2$$

2. Hamiltonian or 'energy'

$$E = \sum_n \phi_n \phi^*_{n+1} + \phi^*_n \phi_{n+1} + \phi^2_n \phi^{*2}_n / 2$$

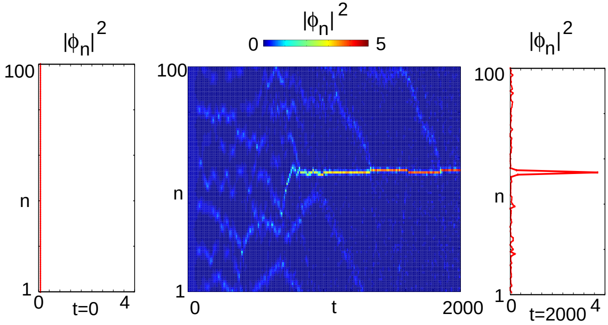
$\rightarrow$ equation of motion $i\dot{\phi}_n = -\frac{\partial E}{\partial \phi^*_n}$

Hamiltonian= coupling-contribution + quartic contribution

Numerical integration of the discrete nonlinear Schrödinger equation

$$i\dot{\phi}_n + \phi_{n+1} + \phi_{n-1} + |\phi_n|^2\phi_n = 0$$

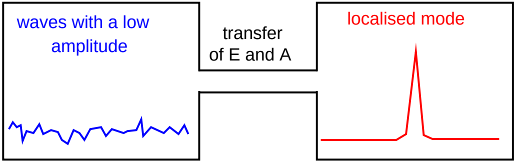
1024 oscillators with periodic boundary condition $\phi_n(t=0) \approx 0.3$



merging solitary waves → localized structures

Thermodynamic model:

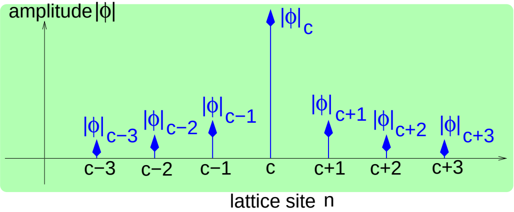
Low-amplitude waves
exchanging energy and particles with a
localised mode



The localised mode:

The maximum of energy for a given norm A:

$$dE - \Omega dA = 0$$

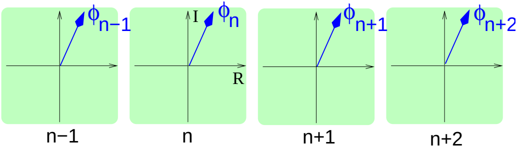


- Ω is the frequency $\phi_n(t) = \phi_n(0)\exp(i\Omega t)$
- High amplitude limit: $E \approx A^2/2$, $\Omega \approx A$

The coupling-energy $E_2 = \sum_n \phi_n \phi^*_{n+1} + \phi^*_n \phi_{n+1}$

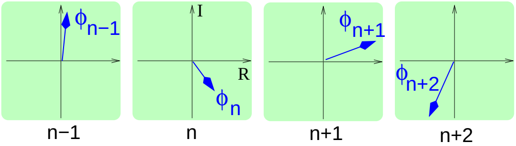
homogeneous mode
$\phi_n = \phi_0$

$E_2 = 2A$



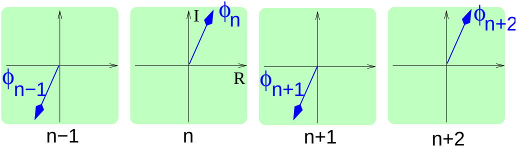
independent
random phases

$E_2 \approx 0$



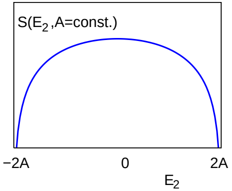
Mode $k = \pi$
$\phi_n = \phi_0 \exp(i\pi n)$

$E_2 = -2A$



Entropy of waves with low amplitudes

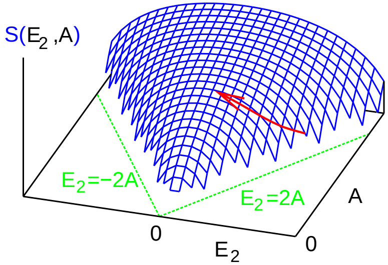
$$S(E_2, A) = N \ln \frac{4A^2 - E_2^2}{4AN}$$



Maximum of the entropy for A fixed is at $E_2 = 0$

The entropy increases when the energy decreases:

The surplus of energy is stored in localised modes



BR, PRE 69, 016618 (2004)

Growth or destruction of localised modes:

High-amplitude peaks grow, low-amplitude peaks decay

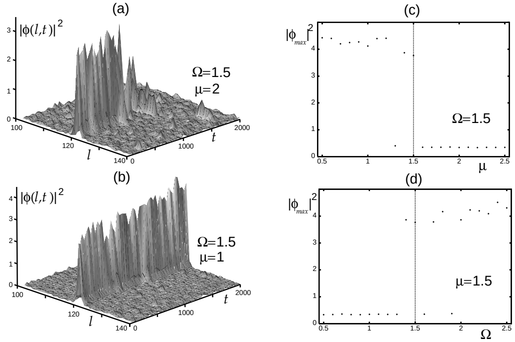
Localised mode:

$dE/dA = \Omega$ (phase frequency)

Isentrope for low-amplitude waves:

$dE/dA|_{S=const.} = \mu$ (chemical potential)

- $\Omega < \mu$:
  the peak decays,
  the particle-effect
  prevails
- $\Omega > \mu$:
  the peak grows,
  the energy-effect
  prevails



BR, EPL 78, 26001 (2007)

BR, Physica D 238, 2067 (2009)

Majda-McLaughlin-Tabak (MMT) model

for weakly nonlinear waves

$$\left(i\frac{\partial}{\partial t} - \mathcal{L}\right)\psi(x,t) \quad = \quad \lambda\psi(x,t)|\psi(x,t)|^2$$

- complex wave amplitude $\psi(x,t)$
- linear operator $\mathcal{L}\exp(ikx) = \omega_k \exp(ikx)$,
- dispersion $\omega_k = \sqrt{|k|}$.

(Fourier modes $a_k = \int_{-L/2}^{L/2} \psi(x,t)\exp(-ikx)dx/\sqrt{2\pi}$
Large system size L with periodic boundary conditions)

A.J. Majda, D.W. McLaughlin, E.G. Tabak, J. Nonlinear Sci. 6, 9 (1997),

Conserved quantities of the MMT equation

- Hamiltonian or 'energy'

$$E = \sum_k \omega_k |a_k|^2 + (\lambda/2) \int_{-L/2}^{L/2} |\psi|^4 dx$$

- waveaction

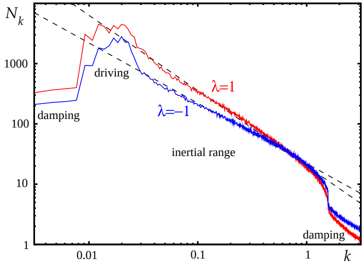
$$N = \sum_k |a_k|^2$$

- momentum

$$P = \sum_k k |a_k|^2$$

Damped and driven MMT equation

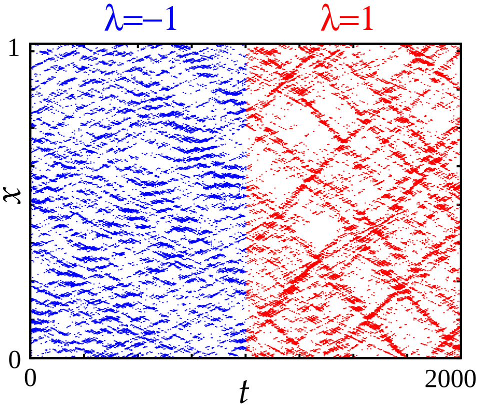
- Driving force at moderate wavenumbers
- Damping at high and at low wavenumbers
- $N_k = \langle |a_k|^2 \rangle$



- $\lambda = -1$: Kolmogorov-Zakharov spectrum $N_k \sim k^{-1}$
  $\rightarrow$ wave turbulence
- $\lambda = 1$: Steeper spectrum $N_k \sim k^{-1.25}$
  $\rightarrow$ unknown mechanism of turbulence

Contour plot of regions with high amplitudes: Switching the sign $\lambda = -1$ from to $\lambda = 1$

causes an instability by negative Landau-damping

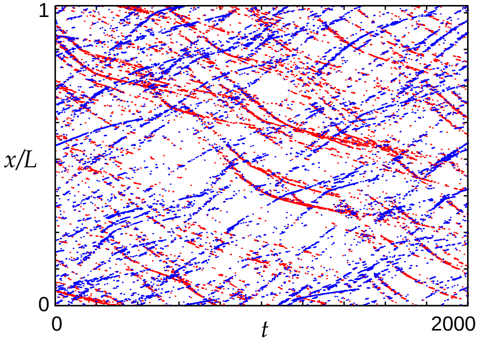


Wave turbulence - Coherent structures

A.C. Newell, B.R., V.E. Zakharov, PRL 108, 194502 (2012)

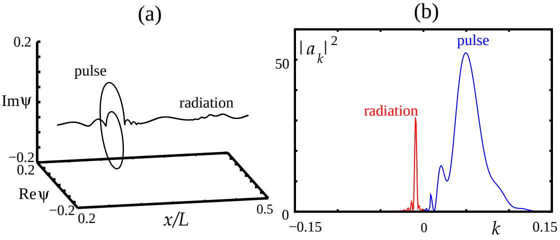
Formation of coherent structures for $\lambda = 1$:

A gas of solitary waves



Pattern of solitary waves ('pulses') with high positive or negative momenta

Radiation: Resonant driving of linear waves



linear wave: driving force by the pulse:

$$i\dot{a}_k - \omega_k a_k = |T_k| \exp(-i\Lambda_k t)$$

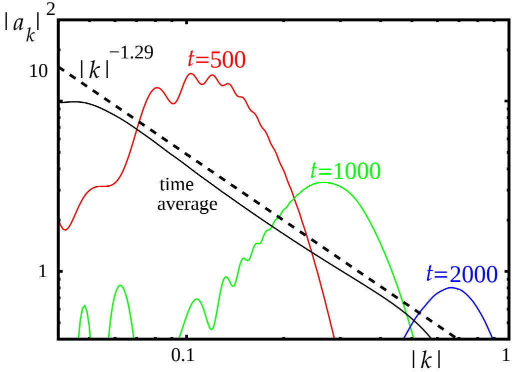
with

$$T_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi^{(pulse)} |\psi^{(pulse)}|^2 \exp(-ikx) dx$$

Analytic solution of the MMT spectrum

- Solve coupled equations for pulse and radiation
- Time-average of the pulse yields the spectrum

$$\langle |a_k^{(pulse)}|^2 \rangle_{time} \sim k^{-2+1/\sqrt{2}} \sim k^{-1.29}$$



B.R., A.C. Newell, V.E. Zakharov, PRL 103, 074502 (2009)

Conclusions

- Peaks emerge in the an environment of negative-temperature waves in the DNLS
- Similar coherent structures emerge in non-thermal distributions in statistically stationary nonequilibrium processes

Spin-Model:
B.R., A.C. Newell, PRL 87, 054102 (2001)
B.R., A.C. Newell, Physica D 184, 162 (2003)
DNLS:
B.R., PRE 69, 016618 (2004)
B.R., EPL 78, 26001 (2007)
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BR, Physica D 238, 2067 (2009)
MMT-Model:
B.R., A.C. Newell, V.E. Zakharov, PRL 103, 074502 (2009)
A.C. Newell, B.R., V.E. Zakharov, PRL 108, 194502 (2012)